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Factorization of heavy-to-light form factors in soft-collinear effective theory

M. Beneke[a] and Th. Feldmann[b]

[a]*Institut für Theoretische Physik E, RWTH Aachen D–52056 Aachen, Germany*

[b]*CERN, Theory Division, CH-1211 Geneva, Switzerland*

Abstract

Heavy-to-light transition form factors at large recoil energy of the light meson have been conjectured to obey a factorization formula, where the set of form factors is reduced to a smaller number of universal form factors up to hard-scattering corrections. In this paper we extend our previous investigation of heavy-to-light currents in soft-collinear effective theory to final states with invariant mass Λ^2 as is appropriate to exclusive B meson decays. The effective theory contains soft modes and two collinear modes with virtualities of order $m_b\Lambda$ ('hard-collinear') and Λ^2. Integrating out the hard-collinear modes results in the hard spectator-scattering contributions to exclusive B decays. We discuss the representation of heavy-to-light currents in the effective theory after integrating out the hard-collinear scale, and show that the previously conjectured factorization formula is valid to all orders in perturbation theory. The naive factorization of matrix elements in the effective theory into collinear and soft matrix elements may be invalidated by divergences in convolution integrals. In the factorization proof we circumvent the explicit regularization of endpoint divergences by a definition of the universal form factors that includes hard-collinear, collinear and soft effects.

1 Introduction

The purpose of this paper is to develop further the theory of exclusive B decays to light energetic mesons. We are specifically interested in "hard-spectator scattering", i.e. scattering mechanisms that involve the light antiquark in the $\bar{B}$ meson. The soft external quark line is one of the crucial differences to the standard situation of hard exclusive processes [1] involving only light hadrons, where all external lines carry large momentum. A consequence of this difference is that exclusive B meson decays involve two hard scales, m_b^2 and $m_b\Lambda$, where Λ is of order of the strong interaction scale.

Hard-spectator scattering is an important ingredient in QCD factorization for non-leptonic B decays to charmless final states [2, 3], and is even more important in the so-called PQCD approach [4]. A better understanding of spectator interactions is needed to justify the factorization hypotheses of the two approaches to all orders in perturbation theory and to leading order in the heavy-quark limit. However, even apparently simpler processes such as the semi-leptonic decay $B \to \pi l\nu$ at large momentum transfer to the pion are currently not completely understood. The one exception is $B \to \gamma l\nu$, which has received much attention recently [5, 6, 7, 8]. A factorization formula of the schematic form $F = T \star \phi_B$, where the star-product denotes convolution of a hard-scattering kernel with the B meson light-cone distribution amplitude, has now been shown to be valid to all orders in perturbation theory [7, 8]. In this paper we consider $B \to \pi l\nu$ (heavy-to-light form factors) at large pion energy. A summary of the results has already been given in [9].

A straightforward extension of the results for $B \to \gamma l\nu$ decay to the $B \to \pi$ form factors relevant to semi-leptonic decays fails. If one writes the form factors as $\phi_\pi \star T \star \phi_B$ in analogy to the $\pi \to \pi$ transition form factor at large momentum transfer, one finds that the convolution integrals do not converge at their endpoints. In other words, the form factors receive leading contributions from momentum configurations where some partons in the pion appear to have small momentum [10, 11]. In [12] the factorization formula

$$F_i = C_i\, \xi_\pi + \phi_B \star T_i \star \phi_\pi \tag{1}$$

has been conjectured for the three Lorentz invariant $B \to \pi$ form factors and shown to be valid at order α_s. The additional term $C_i\, \xi_\pi$ involves a short-distance coefficient and a single "soft form factor" ξ_π, which obeys the large-recoil symmetries [13]. Factorization for the $B \to \pi$ form factors is more complicated than for both, the $\pi \to \pi$ and $B \to D$ form factors. At large momentum transfer soft interactions cancel in the $\pi \to \pi$ transition at leading power. The remaining hard and collinear effects are factored into convolutions as in the second term on the right-hand side of (1). When both mesons are heavy, such as in $B \to D$, collinear effects are irrelevant. The remaining hard and soft interactions factor into a short-distance coefficient and the Isgur-Wise form factor [14] similar to the first term on the right-hand side of (1). The $B \to \pi$ form factor, however, involves hard, collinear and soft effects. Furthermore, due to the presence of several scales one must distinguish short- and long-distance collinear effects, as we discuss in more detail below.

A separation of all these effects and an operator definition of the various short- and long-distance quantities in (1) to all orders in α_s and to leading order in $1/m_b$ has not

Table 1: Terminology for the various momentum modes relevant to exclusive B decays. The momentum components are given as $(n_+p, p_\perp, n_-p)$, but mass dimension has to be restored by multiplying appropriate factors of m_b. Two different terminologies for the same momentum modes are used in the literature. In physical units λ is of order $(\Lambda/m_b)^{1/2}$, where Λ is the strong-interaction scale.

Momentum scaling	Terminology I ([16],[17])	Terminology II ([20], this work)
$(1, 1, 1)$	hard	hard
$(\lambda, \lambda, \lambda)$	soft	semi-hard
$(1, \lambda, \lambda^2)$	collinear	hard-collinear
$(\lambda^2, \lambda^2, \lambda^2)$	ultrasoft	soft
$(1, \lambda^2, \lambda^4)$	ultracollinear	collinear

yet been achieved. A complication was pointed out in [15], where it was shown that superficially sub-leading interactions in $1/m_b$ contribute to the $B \to \pi$ form factors at leading power. Some of the previous arguments [16, 17] to justify or extend some aspects of (1) must therefore be revised. The form factors have been reconsidered in [15] in the framework of soft-collinear effective field theory (SCET) [16, 18] and the structure of the formula (1) was seen to emerge. However, in [15] a technical definition of "factorizable" and "non-factorizable" terms has been adopted that does not correspond to the usual notions, so that the issues of endpoint singularities and convergence of convolution integrals could not be clarified. Below we extend SCET in the position-space formulation [17, 19] to cover the case of exclusive decays, where the external collinear lines have invariant mass of order Λ^2 as appropriate for exclusive decays. To obtain the factorization formula (1) we match SCET to an effective theory from which short-distance collinear modes with virtuality $m_b\Lambda$ are removed. This point was first addressed in [20] and the formalism has been worked out to the extent that a factorization theorem for $B \to \gamma l \nu$ was established [8].

The momentum modes relevant to the factorization of form factors $\langle M(p')|\bar{q}\Gamma b|\bar{B}(p)\rangle$, where M is a light meson or a photon with momentum of order m_b, and Γ is a Dirac matrix, are summarized in Table 1. In general we decompose a momentum as

$$p^\mu = (n_+p)\,\frac{n_-^\mu}{2} + p_\perp^\mu + (n_-p)\frac{n_+^\mu}{2}, \tag{2}$$

where $n_\pm^\mu$ are two light-like vectors, $n_+^2 = n_-^2 = 0$ with $n_+n_- = 2$. The reference directions $n_\pm$ are chosen such that the energetic massless external lines have n_+p of order m_b. As indicated in Table 1 two different terminologies have been used in the literature which has

been the cause of some confusion. Since in this paper we will construct an effective theory for modes with virtuality Λ^2 only, we will use the second terminology. The effective theory then contains soft and collinear modes in agreement with the standard QCD terminology. For power counting we define the scaling parameter λ to be of order $(\Lambda/m_b)^{1/2}$. This differs from the convention in [20] where λ is of order Λ/m_b.

The existence of the various modes follows from the assumption that the external momenta of scattering amplitudes for exclusive B decays at large momentum transfer are soft or collinear.[1] One finds the three characteristic virtualities m_b^2, $m_b\Lambda$ and Λ^2 by combining external momenta. For instance, m_b^2 is obtained by adding and squaring a heavy quark and a collinear momentum, or by squaring the heavy quark momentum. The intermediate virtuality is typical for interactions of collinear gluons or light quarks with soft gluons or light quarks, while Λ^2 arises in the self-interactions of collinear or soft modes.

Soft-collinear effective theory as defined in [16, 17] is the effective theory obtained after integrating out hard modes of virtuality m_b^2. This theory still contains two types of soft modes, called "semi-hard" (virtuality of order $m_b\Lambda$) and "soft". The semi-hard modes can be integrated out perturbatively, but it appears that semi-hard loop integrals always vanish in dimensional regularization [17], so semi-hard modes can be ignored in practice. The theory also contains two types of collinear modes, called "hard-collinear" and "collinear" according to their virtuality. Although each one of these two has been discussed in previous applications of SCET, the simultaneous presence of two distinct collinear modes has not been considered in much detail up to now. The reason for this is that previous applications of SCET to semi-inclusive B decay, such as $B \to X_s\gamma$ near maximal photon energy [18], and to $B \to \gamma l\nu$ (at leading order in $1/m_b$) are sensitive only to hard-collinear modes [7, 8]. One can therefore match SCET directly to the standard heavy quark effective theory, which contains only soft modes. On the other hand, in the exclusive decay $B \to D\pi$ [3, 21] or hard exclusive scattering of light hadrons [22] the effects of hard-collinear and soft modes cancel almost trivially at leading power in the power expansion. The effective theory at leading power can then be formulated entirely in terms of collinear modes.

The outline of this paper is as follows: in Section 2 we study a scalar integral, which would be relevant to $B \to \gamma l\nu$ decay at sub-leading order in $1/m_b$, using the method of expanding by regions [23]. We demonstrate with this example that separate contributions from hard-collinear and collinear loop momentum must be included to reproduce the integral in full "QCD". We shall also find that the collinear and soft contributions are not well-defined individually in dimensional regularization. The interpretation of these additional divergences provides an important clue to the problem of endpoint divergences. In the context of effective field theory the additional divergences show that the matrix elements in the effective theory of soft and collinear fields do not factorize (naively) into a product of soft and collinear matrix elements as one might have concluded from the decoupling of soft and collinear fields in the Lagrangian.

[1] For the heavy quark line we write $p_b = m_b v + r$, where v is the B meson velocity. The statement that the external heavy quark is soft refers to the fact that after this decomposition the residual momentum r is soft.

In Section 3 we turn our attention to the representation of the heavy-to-light current in the effective theory with the hard-collinear scale removed. We integrate out the hard-collinear modes in tree graphs by solving the classical field equations for the hard-collinear quark and gluon fields. Despite the complex branchings of the relevant trees, the solution can be found by choosing a special gauge for the calculation and reconstructing the complete result through gauge invariance. The first term with a non-vanishing $\langle\pi|\ldots|\bar{B}\rangle$ matrix element in the expansion of the current is λ^3 suppressed, which explains the $1/m_b^{3/2}$ suppression of heavy-to-light form factors at large recoil. The calculation also shows that the effective operator is highly non-local, implying convolutions in two light-like directions. The convolutions are divergent, as expected, but we also find that quark-antiquark-gluon amplitudes in the B meson and in the light meson contribute at leading power, which is a new feature of heavy-to-light transitions. At the end of this section we briefly discuss hard-collinear quantum corrections to determine the general form of operators and short-distance kernels in the effective theory of soft and collinear modes.

The existence of divergent convolutions signals that heavy-to-light form factors do not factorize straightforwardly. In Section 4 we return to the factorization formula (1), and show that it is indeed valid to all orders in the strong coupling and to leading power in $1/m_b$. We shall define the universal form factor ξ_π as a matrix element in SCET before integrating out hard-collinear effects. We then show that the terms not contained in this definition factorize into convolutions of light-cone distribution amplitudes with convergent integrals after integrating out hard-collinear modes. The proof of convergence relies on power counting, boost invariance, and the correspondence of collinear and soft endpoint divergences through soft-collinear factorization. We conclude in Section 5.

2 The scalar "photon" vertex

The purpose of this section is to demonstrate by the example of a specific Feynman integral that the distinction of hard-collinear and collinear modes has a technical meaning. We shall also see how the factorization of collinear and soft modes introduces "endpoint singularities" in the longitudinal integrations, and how the singularities are related to cancel in the sum of all terms. Finally, we sketch how the diagrammatic result would be interpreted in the context of effective field theory.

We consider the scalar integral

$$I = \int [dk] \, \frac{1}{[(k-l)^2][k^2-m^2][(p'-k)^2-m^2]}, \tag{3}$$

which occurs as a one-particle-irreducible subgraph in the correction to the radiative decay $\bar{B} \to \gamma l \nu$ shown in Figure 1. We keep a "light quark" mass m, which will take the role of Λ as the infrared scale of our problem. We define the integration measure as

$$[dk] \equiv \frac{(4\pi)^2}{i} \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^\epsilon \frac{d^dk}{(2\pi)^d} = \mu^{2\epsilon} e^{\epsilon\gamma_E} \, \frac{d^dk}{i\pi^{d/2}} \qquad (d = 4-2\epsilon). \tag{4}$$

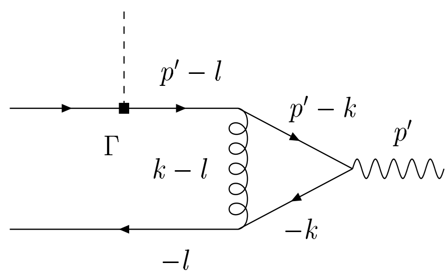

Figure 1: Photon vertex correction to $\bar{B} \to \gamma l \nu$. Γ denotes the weak $b \to u$ decay vertex. We consider the corresponding vertex integral with all lines simplified to scalar propagators and all vertex factors set to 1.

The integral I is ultraviolet and infrared finite, but dimensional regularization will be needed to construct the expansion. The $+i\epsilon$ prescription for the propagators is understood.

The external momenta of the vertex subgraph are: a collinear "photon" momentum $p' = (n_+ p', p'_\perp, n_- p') \sim (1, 0, 0)$ with $p'^2 = 0$; a soft "light quark" momentum $l \sim (\lambda^2, \lambda^2, \lambda^2)$ with $l^2 = m^2$; a hard-collinear "light quark" momentum $p' - l \sim (1, \lambda^2, \lambda^2)$ with virtuality λ^2. The two invariants are $2p' \cdot l \sim \lambda^2$ and $m^2 \sim \lambda^4$, so I must be a function of the small dimensionless ratio $m^2/(2p' \cdot l)$. A straightforward calculation gives

$$I = \frac{1}{2p' \cdot l} \left(\mathrm{Li}_2 \left(-\frac{2p' \cdot l}{m^2} \right) - \frac{\pi^2}{6} \right) = -\frac{1}{2p' \cdot l} \left\{ \frac{1}{2} \ln^2 \frac{m^2}{2p' \cdot l} + \frac{\pi^2}{3} + \ldots \right\}, \tag{5}$$

where after the second equality we neglected higher-order terms in $m^2/(2p' \cdot l)$. Below we shall reproduce the first term in the expansion by identifying the relevant momentum configurations.

2.1 Expansion by momentum regions

We construct the expansion of I by identifying the momentum configurations that give non-vanishing contributions to the integral in dimensional regularization and by expanding the integrand in each region [23]. This method, originally developed for a non-relativistic expansion and the construction of non-relativistic effective theory, has also been applied to integrals with collinear external lines [24]. Here we need a variant for integrals with collinear and soft external lines.

To find the relevant momentum regions we first assume that the loop momentum scales as $k \sim (\lambda^n, \lambda^n, \lambda^n)$ for some n and expand the integrand accordingly. For instance, if k is hard, $n = 0$, we find integrals of the type

$$\int [dk] \, \frac{1}{[k^2]^a [-2p' \cdot k]^b} \times \text{polynomial}, \tag{6}$$

which vanish in dimensional regularization, since the only possible invariant $p'^2 = 0$. This is not surprising, because there is no external invariant of order 1. Proceeding for different

n, the result is that only the soft momentum region, $n = 2$, contributes. The corresponding integral will be calculated below.

Since there are external lines with large momentum and small virtuality, we should also consider loop momentum configurations, where n_+k is the largest component. That is, we take $n_+k \sim \lambda^n$ and $k^2 \sim \lambda^{2m}$ with $m < n$, expand the integrand, and determine the integrals that do not vanish.[2] The result is two non-vanishing contributions, one from $n_+k \sim 1$ and $k^2 \sim \lambda^2$, which we identify as hard-collinear, and the other from $n_+k \sim 1$ and $k^2 \sim \lambda^4$, which we call collinear, see Table 1. Regions with $k^2 < \lambda^4$ do not appear due to the internal masses $m^2 \sim \lambda^4$. We will now verify (at leading order) that the sum of the three regions constructs the expansion of our integral I.

The hard-collinear region. Expanding the propagators systematically the leading hard-collinear integral is

$$\begin{aligned} I_{hc} &= \int [dk] \, \frac{1}{[k^2 - n_+kn_-l][k^2][k^2 - n_+p'n_-k]} \\ &= -\frac{1}{2p' \cdot l} \left\{ \frac{1}{\epsilon^2} - \frac{1}{\epsilon} \ln \frac{2p' \cdot l}{\mu^2} + \frac{1}{2} \ln^2 \frac{2p' \cdot l}{\mu^2} - \frac{\pi^2}{12} \right\}. \end{aligned} \tag{7}$$

The expansion has rendered the integral infrared divergent. If we perform the n_-k integration by contour methods, the $k_\perp$ integral is divergent for $k_\perp \to 0$ (physically, $k_\perp \ll \lambda$) for any n_+k, but the n_+k integral converges at fixed $k_\perp$. The double pole originates from $n_\pm k \to 0$, $k_\perp \to 0$ simultaneously.

The collinear region. In this region the "light quark" propagators with momenta $p' - k$ and $-k$ are collinear and have virtuality of order λ^4. The "gluon" propagator is hard-collinear with virtuality λ^2. One finds that the collinear and soft integrals are not well-defined separately in dimensional regularization. This also occurred in previous applications of the method of expansion by regions to collinear integrals [24], and is related to the fact the dimensional regulator is attached to the transverse momentum components. If additional divergences arise from the n_+k or n_-k integrations, they may not be regularized. As in [24] we introduce an additional "analytic" regularization by substituting

$$\frac{1}{[(k-l)^2]} \to \frac{[-\nu^2]^\delta}{[(k-l)^2]^{1+\delta}}, \tag{8}$$

where ν is a parameter with mass dimension one. The leading collinear integral is

$$I_c = \int [dk] \, \frac{[-\nu^2]^\delta}{[-n_+kn_-l]^{1+\delta}[k^2 - m^2][k^2 - m^2 - 2p' \cdot k]}. \tag{9}$$

[2]Some integrals vanish independent of any regularization, because all poles lie in one of the complex half-planes. Other integrals vanish only, because we assume a regularization that does not introduce an additional scale into the integral.

The integral can be done with standard methods, but it will be useful to obtain an intermediate result, where only the n_-k integration is performed. The variable n_+p' is related to the energy of the "photon", so $n_+p' > 0$. We then close the contour in the lower half plane and pick up the pole at $(-k_\perp^2 + m^2 - i\epsilon)/n_+k$ for $0 < n_+k < n_+p'$. (In our convention $k_\perp^2$ is negative.) This gives

$$\begin{aligned} I_c &= -\frac{1}{2p'\cdot l}\left(\frac{\nu^2}{2p'\cdot l}\right)^\delta \int_0^{n_+p'} \frac{dn_+k}{n_+k}\left(\frac{n_+p'}{n_+k}\right)^\delta \left(\mu^2 e^{\gamma_E}\right)^\epsilon \int \frac{d^{d-2}k_\perp}{\pi^{d/2-1}}\,\frac{-1}{k_\perp^2 - m^2} \\ &= -\frac{1}{2p'\cdot l}\left(-\frac{1}{\delta} + \ln\frac{2p'\cdot l}{\nu^2}\right)\left(\frac{1}{\epsilon} - \ln\frac{m^2}{\mu^2}\right). \end{aligned} \tag{10}$$

The pole at $\epsilon = 0$ comes from the collinear singularity $k_\perp \to \infty$ (for any n_+k) in the transverse momentum integral. The additional singularity at $\delta = 0$ is an "endpoint divergence", which arises from a singularity at $n_+k \to 0$ for any transverse momentum. This does not correspond to any singularity of the hard-collinear integral. Since n_-k becomes large compared to λ^4, when n_+k becomes small, the endpoint singularity is related to a momentum configuration, where the "quark" with momentum k becomes soft. In particular, since the endpoint divergence occurs for any $k_\perp \sim \lambda^2$ it must be cancelled by a momentum region with $k_\perp \sim \lambda^2$.

Also note that the integral depends non-analytically on the soft external momentum component n_-l. This is surprising, since we would have expected factorization of soft and collinear modes, so that the collinear integrals could depend only analytically on n_-l, and the soft integrals could depend only analytically on n_+p'. Indeed, this would be the case in dimensional regularization, where the factor $1/n_-l$ in (9) could be pulled out of the integral. However, the integral is not well-defined in dimensional regularization. The breakdown of naive collinear-soft factorization is hence a consequence of the need to introduce a different regularization, here chosen as analytic.

The soft region. In this region the "gluon" propagator and the "light quark" propagator with momentum $-k$ are soft and have virtuality of order λ^4. The "light quark" with momentum $p' - k$ is hard-collinear with virtuality λ^2. The leading soft integral is

$$I_s = \int [dk]\,\frac{[-\nu^2]^\delta}{[(k-l)^2]^{1+\delta}[k^2 - m^2][-n_+p' n_-k]}. \tag{11}$$

Here we perform the n_+k integral first. Assuming $n_-l > 0$, we close the contour in the lower half plane and pick up the pole at $(-k_\perp^2 + m^2 - i\epsilon)/n_-k$ for $0 < n_-k < n_-l$. This gives

$$\begin{aligned} I_s &= -\frac{1}{2p'\cdot l}\int_0^{n_-l}\frac{dn_-k}{n_-k}\left(\frac{n_-l}{n_-k}\right)^{-\delta}\left(\mu^2 e^{\gamma_E}\right)^\epsilon \int \frac{d^{d-2}k_\perp}{\pi^{d/2-1}}\,\frac{\nu^{2\delta}}{(m^2\left[1 - \frac{n_-k}{n_-l}\right]^2 - k_\perp^2)^{1+\delta}} \\ &= -\frac{1}{2p'\cdot l}\left(\frac{\mu^2 e^{\gamma_E}}{m^2}\right)^\epsilon \Gamma(\epsilon)\left(\frac{m^2}{\nu^2}\right)^{-\delta}\,\frac{1}{\delta}\,\frac{\Gamma(\delta+\epsilon)\Gamma(1-2\delta-2\epsilon)}{\Gamma(\epsilon)\Gamma(1-\delta-2\epsilon)}. \end{aligned} \tag{12}$$

There is a singularity for $k_\perp \to \infty$ for any n_-k. The pole at $\delta = 0$ is an endpoint divergence from $n_-k \to 0$ for any $k_\perp$. This implies that n_+k becomes large for fixed $k_\perp \sim \lambda^2$, and hence the "quark" with momentum k becomes collinear. In the soft region the transverse momentum and longitudinal momentum integrals do not factorize, and there is also a divergence when $k_\perp \to \infty$ and $n_-k \to 0$ simultaneously, which corresponds to the double pole in the hard-collinear integral.

Since we did not regularize the hard-collinear contribution analytically, the correct procedure is to expand first in δ and then in ϵ. In fact the pole in δ cancels with the collinear contribution before expanding in ϵ. However, performing both expansions to compare with (10) we obtain

$$I_s = -\frac{1}{2p'\cdot l}\left(\left[\frac{1}{\delta} - \ln\frac{m^2}{\nu^2}\right]\left[\frac{1}{\epsilon} - \ln\frac{m^2}{\mu^2}\right] - \frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{m^2}{\mu^2} - \frac{1}{2}\ln^2\frac{m^2}{\mu^2} + \frac{5\pi^2}{12}\right) \qquad (13)$$

The expansion in δ has generated a double pole in ϵ.

Adding up. The singularity in δ cancels in the sum of the collinear and soft integral

$$I_c + I_s = -\frac{1}{2p'\cdot l}\left(-\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{2p'\cdot l}{\mu^2} - \ln\frac{2p'\cdot l}{\mu^2}\ln\frac{m^2}{\mu^2} + \frac{1}{2}\ln^2\frac{m^2}{\mu^2} + \frac{5\pi^2}{12}\right). \qquad (14)$$

Finally adding to this the hard-collinear contribution (7), the singularity in ϵ also cancels, and we obtain

$$I_c + I_s + I_{hc} = -\frac{1}{2p'\cdot l}\left(\frac{1}{2}\ln^2\frac{2p'\cdot l}{m^2} + \frac{\pi^2}{3}\right), \qquad (15)$$

in agreement with the expansion (5) of the full integral. We conclude that in general hard-collinear, collinear and soft momentum regions must be considered. In the scalar integral (3) all three regions contribute already to the leading term in the expansion. Semi-hard modes with scaling $(\lambda, \lambda, \lambda)$ are not needed in this calculation, since the corresponding integrals are scaleless.

In QCD the photon-vertex integral contains a numerator proportional to n_-k which suppresses the collinear region by a factor of λ^2 relative to the hard-collinear and soft region. For this reason it is sufficient to consider only hard-collinear and soft configurations in the factorization theorem for $B \to \gamma l \nu$ at leading power in $1/m_b$, as has been done in [6, 7, 8]. Hard-collinear modes are perturbative and can be integrated out, resulting in hard-scattering kernels. Soft and collinear modes have virtuality $\lambda^4 \sim \Lambda^2$, and cannot be treated in perturbation theory. The $1/m_b$ suppression of the collinear contribution in QCD implies that the hadronic structure of the photon is a sub-leading effect in $B \to \gamma l \nu$ decay.

2.2 Off-shell regularization

The scalar integral (10) has recently been discussed in [25], however with $m = 0$ and the external collinear and soft lines off-shell, $l^2 \equiv -L^2 = n_+ l n_- l \sim \lambda^4$, and $(p')^2 \equiv -(P')^2 = n_+p'n_-p' \sim \lambda^4$. It is instructive to discuss the difference to the case above.

The hard-collinear integrals are identical as they should be, because the two integrals differ only at the small scale λ^4. The collinear contribution is now given by

$$I_c' = -\frac{1}{2p'\cdot l}\left(\frac{\nu^2}{2p'\cdot l}\right)^{\delta}\int_0^1 \frac{du}{u^{1+\delta}}\,\left(\mu^2 e^{\gamma_E}\right)^{\epsilon}\int\frac{d^{d-2}k_\perp}{\pi^{d/2-1}}\,\frac{-1}{k_\perp^2-u(1-u)(P')^2}, \tag{16}$$

with $n_+k = un_+p'$. This is to be compared to the first line of (10). When the transverse momentum integral is taken, it supplies a factor $[u(1-u)]^{-\epsilon}$, so the integral does appear to be defined in dimensional regularization ($\delta = 0$). However, at fixed transverse momentum there is still an endpoint divergence as $u \to 0$, which is not regularized dimensionally. Hence the integral has the same divergence as $k_\perp \to \infty$ and the same endpoint divergence as in the on-shell, massive case. In particular, the endpoint divergence cancels again with a soft endpoint divergence, at any transverse momentum.

The off-shell integral exhibits a new type of divergence, when $k_\perp \to 0$ and $n_+k \to 0$ simultaneously. Since $k_\perp \to 0$ this must be related to a new momentum region with transverse momentum smaller than λ^2. A similar comparison of the soft integral shows that it possesses a singularity for $k_\perp \to 0$ and $n_-k \to 0$ in addition to those already discussed. In [25] it is shown that these extra divergences are compensated by a "soft-collinear" contribution, where the loop momentum scales as $(\lambda^2, \lambda^3, \lambda^4)$, and that the sum of all four contributions reproduces the expansion of the full integral. The various divergences and their relations are summarized in Figure 2.

Applied to effective theories of QCD, where $\lambda^2 \sim \Lambda$, the various factorization steps shown in the Figure correspond to matching SCET onto an effective theory of only soft and collinear modes, and the factorization of soft and collinear modes within the low-energy theory. The third factorization step is needed only for the off-shell, massless case, and does not have an equivalent in QCD, since it concerns the separation of transverse momentum scales below Λ. Contrary to QCD, where we suppose that non-perturbative dynamics provides a universal infrared cut-off of order Λ, the off-shell regularization of a massless Feynman integral does not prevent sensitivity to arbitrarily small loop momenta.[3] This need not hinder the use of the off-shell regularization and the introduction of soft-collinear modes as a technical construct. This point of view has been taken in [26]. However, it is clear from (16) that the breakdown of naive soft-collinear factorization is related to endpoint divergences that already occur at transverse momenta of order λ^2, and is *per se* unrelated to the existence of soft-collinear modes. In dimensional regularization soft-collinear modes must be introduced as a way to recover the non-analytic dependence of the integral on $n_+p'n_-l$. From a physical point of view, however, it is sufficient to describe the low-energy physics in terms of collinear and soft modes only.

It is important to keep in mind that matrix elements in the low-energy effective theory cannot be naively factorized into soft and collinear matrix elements. We may always factorize soft and collinear contributions in a perturbative Feynman integral with an appropriate regulator (as done above), but when λ^2 is set to Λ, one must examine whether factorization

[3] It is very likely that with more loops, more modes of successively smaller virtuality must be introduced, with no lower limit on the virtuality as the number of loops increases.

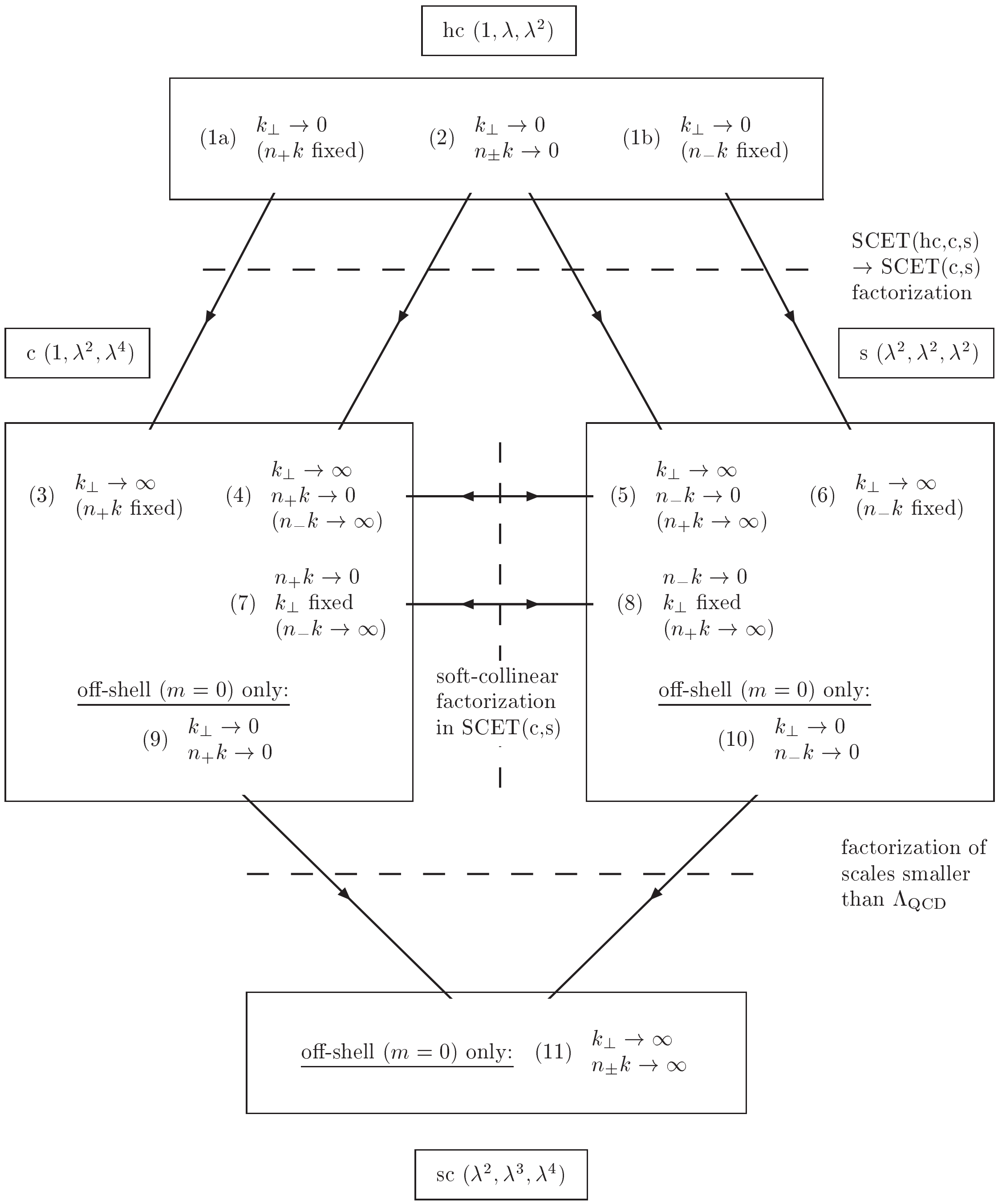

Figure 2: Divergence structure of the integral (3) and its off-shell, massless version when expanded by regions. The arrows indicate the divergences in different regions that are related and cancel each other. The dashed lines mark the various factorization steps.

occurs at weak coupling. If it does not, factorization of soft and collinear modes cannot be implemented perturbatively in QCD. If endpoint divergences can be shown to be absent to all orders in perturbation theory, so that no regularization is required, this indicates that naive soft-collinear factorization is valid.

2.3 Interpretation of the result

Hard-collinear modes in SCET. The toy integral clarifies that SCET, defined as the effective theory after integrating out hard modes, contains two collinear modes with different virtuality. If SCET is formulated with a single collinear quark and gluon field, the collinear fields cannot be assigned an unambiguous scaling law, and power counting is no longer manifest in the vertices of the effective theory,[4] unless one of the two collinear modes is irrelevant for a specific process. An alternative is to introduce separate hard-collinear and collinear fields in SCET. The corresponding Lagrangian can be taken as a starting point for the second matching step, in which hard-collinear modes are integrated out and SCET is reduced to an effective Lagrangian for soft and collinear modes only. This formulation will be used in some of the technical steps in the following two sections.

A comment is necessary on the transverse momentum scaling of hard-collinear modes. When a soft and collinear momentum combine to a hard-collinear fluctuation of virtuality λ^2, the hard-collinear transverse momentum must be of order λ^2 by momentum conservation. This is the case for the hard-collinear propagators in the soft and collinear contributions to the toy integral (see the upper row of Figure 3). However, in hard-collinear loops the transverse momentum is of order λ, as one can easily verify from the location of poles of the hard-collinear integrand. Assuming $k_\perp \sim \lambda^2$ would make all hard-collinear loop integrals vanish, since the integrals would have to be expanded in $k_\perp^2$. This would obviously fail to reproduce the expansion of the exact integral. We therefore assign the generic scaling $(1, \lambda, \lambda^2)$ to hard-collinear modes, as given in Table 1. Non-generic scaling in tree subgraphs is not particular to the present case of hard-collinear modes. When one integrates out hard heavy quark fluctuations generated by the interaction of near on-shell heavy quarks with hard-collinear or collinear gluons [16], the off-shell modes have momentum $(1, \lambda, 1)$ or $(1, \lambda^2, 1)$, unlike the generic hard momentum $(1, 1, 1)$.

Operator interpretation of the toy integral. We proceed to discuss the three contributions to the toy integral in terms of operators and matrix elements of an effective theory for soft and collinear modes. This discussion will be heuristic, since we abstract from the scalar integral and use QCD terminology, but without making the notation completely explicit.

We imagine that Figure 1 represents a correction to the matrix element $\langle \gamma | J | \bar{q} b \rangle$ of the $b \to u$ transition current between a $\bar{q}b$ state with fixed light quark momentum $n_- l$ and a photon with large energy $E = n_+ p'/2$. The corresponding tree diagram has one hard-collinear line joining the weak vertex to the photon vertex. In the effective theory (of

[4]This also occurs in the standard formulation of non-relativistic effective theory, where the quark and gluon fields do not obey an unambiguous velocity scaling rule.

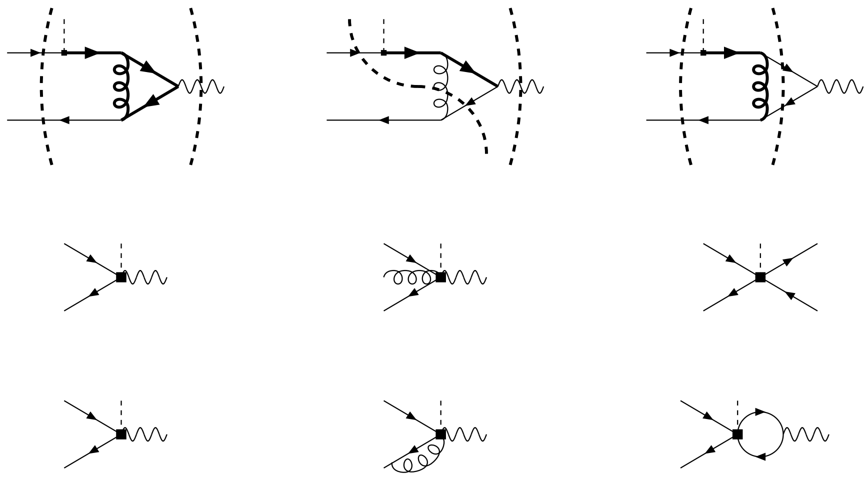

Figure 3: Diagrammatic and operator/matrix element representation of the hard-collinear (left column), soft (middle column) and collinear contribution to the diagram of Figure 1. Each column shows: the original diagram with the hard-collinear subgraph marked by bold-face lines (upper row), and with the dashed line indicating where the graph factorizes into a short-distance and long-distance subgraph; the operator vertex in the effective theory corresponding to the contracted hard-collinear subgraph (middle row); the contribution to the operator matrix element $\langle\gamma|O_i|\bar{q}b\rangle$ corresponding to the original diagram (lower row).

soft and collinear modes) this is written as

$$C_0(E, n_-l)\,{}_{\rm FT}\langle\gamma|[A_\gamma(sn_+)]_c\,[\bar{q}(tn_-)h_v(0)]_s|\bar{q}b\rangle(E, n_-l). \tag{17}$$

The symbol ${}_{\rm FT}\langle\ldots\rangle$ means that a Fourier transform of the matrix element with respect to the position arguments of the fields is taken, with (E, n_-l) the variables conjugate to (sn_+, tn_-). The index on products of fields indicates whether they are soft or collinear, and the non-locality of the operator is related to the non-polynomial dependence of the hard-collinear propagator on the momentum component n_-l of the light external quark, and the momentum component n_+p' of the external photon. The matrix element factorizes trivially into

$${}_{\rm FT}\langle\gamma|[A_\gamma(sn_+)]_c|0\rangle(E)\,{}_{\rm FT}\langle 0|[\bar{q}(tn_-)h_v(0)]_s|\bar{q}b\rangle(n_-l). \tag{18}$$

The photon matrix element can be calculated. When the $\bar{q}b$ state is replaced by a $\bar{B}$ meson, the soft matrix element gives the B meson light-cone distribution amplitude. Hence (17) assumes the form of a convolution of a hard-collinear coefficient function with the B meson light-cone distribution function, which reproduces the factorization property of the $B \to \gamma$ transition at leading order in $1/m_b$, and at leading order in α_s [5].

The hard-collinear contribution to the toy integral and its operator interpretation is shown in the left column of Figure 3. When the hard-collinear subgraph is contracted to a "point", the corresponding operator has the same field content as in (17), but with a

different coefficient function $C_1(E, n_- l)$ as a result of the loop integration. We identify C_1 as a 1-loop correction to the hard-scattering kernel. The explicit calculation shows that there is a double pole in $1/\epsilon$, leaving a double-logarithmic dependence on the factorization scale μ.

Consider now the soft contribution (middle column in the Figure). The hard-collinear subgraph has an additional external soft gluon line, so the operator in the effective theory has the structure $[\bar{q} A h_v]_s \, [A_\gamma]_c$ (second line in the Figure). The matrix element in the third line of Figure 3 takes the form

$$ {}_{\mathrm{FT}}\langle \gamma | [A_\gamma(s n_+)]_c | 0 \rangle (E) \int d\omega \, C_2(E, n_- l, \omega) \, {}_{\mathrm{FT}}\langle 0 | [\bar{q}(t_2 n_-) A(t_1 n_-) h_v(0)]_s | \bar{q} b \rangle (n_- l, \omega). \quad (19) $$

The soft matrix element can be identified with a three-particle light-cone distribution amplitude $\phi_{\bar{q}bg}$ of the $\bar{q}b$ state. Comparing this expression to (12), we see that the $n_- k$ integral in (12) corresponds to the integration over ω, while the transverse momentum integral is included in the definition of the light-cone distribution amplitude.

In the conventional hard-scattering formalism the scale dependence of the distribution amplitude would cancel against the scale dependence of a hard-scattering kernel (such as C_1). This cannot be completely correct here, since the ω-integral has an endpoint divergence as $\omega \to 0$, which corresponds to the $1/\delta$ singularity in (12). The associated ν-dependence is not cancelled by a hard-scattering kernel, but by the collinear contribution as seen from the toy example. The existence of an endpoint divergence implies that expression (19) in its entirety has a scale-dependence different from the two matrix elements in the factorized expression. This possibility is not considered in the conventional hard-scattering formalism.[5]

The operator interpretation of the collinear integral is illustrated in the third column of Figure 3. The photon line is not directly connected to the hard-collinear subgraph in this case. Rather the operator that results after contracting the hard-collinear subgraph has field content $[\bar{q} h_v]_s \, [\bar{q} q]_c$ (second line in the Figure). The matrix element (third line) can be written as

$$ {}_{\mathrm{FT}}\langle 0 | [\bar{q}(t n_-) h_v(0)]_s | \bar{q} b \rangle (n_- l) \int_0^1 du \, C_3(E, u, n_- l) \, {}_{\mathrm{FT}}\langle \gamma | [\bar{q}(s_1 n_+) q(s_2 n_+)]_c | 0 \rangle (E, u). \quad (20) $$

This seems to represent the convolution of a hard-scattering kernel C_3 with the two-particle light-cone distribution amplitude of the $\bar{q}b$ state $\phi_{\bar{q}b}$ and the $q\bar{q}$ light-cone distribution amplitude of the photon $\phi^\gamma_{q\bar{q}}$. This is only correct with the understanding that the u-integral is divergent and must be regularized in a way that is consistent with the regularization of the ω-integral in the soft contribution. The additional divergence, which is not related to

[5] In the leading power analysis of $B \to \gamma l \nu$ decay there is no endpoint divergence from the soft contribution of the photon vertex integral [6]. The reason for this is that the corresponding operator is $[\bar{q} n_- A h_v]_s \, [A_\gamma]_c$, which is related to $[\bar{q} h_v]_s \, [A_\gamma]_c$ by gauge invariance. It is in fact included in the gauge-invariant definition of $[\bar{q} h_v]_s \, [A_\gamma]_c$, which contains a path-ordered exponential. In sub-leading power, the gluon can be transverse, and an endpoint divergence appears. This is consistent with the fact that the collinear contribution to the photon vertex is also sub-leading power in QCD.

the renormalization of the conventional light-cone distribution amplitudes, is the endpoint divergence of the n_+k integral in (10). The associated ν-dependence cancels against the ν-dependence of the soft contribution. In general, the distribution amplitudes may themselves depend on the additional regularization, and hence differ from the distribution amplitudes that appear in the hard-scattering formalism.

To summarize this discussion, we distinguish two steps of factorization. In the first step, we integrate out the large-virtuality hard-collinear modes, and represent the result in terms of operators of soft and collinear fields. These operators will be non-local, reflecting the fact that the hard-scattering kernels appear in convolutions rather than as multiplicative factors. The second factorization step refers to the separation of soft and collinear modes *within* the effective theory of soft and collinear modes. In our example, the photon couples only to collinear lines, and the $\bar{q}b$ state couples only to soft lines, so we would expect the effective theory matrix elements to factorize into a matrix element of collinear fields between the photon and the vacuum, and a matrix element of soft fields between the vacuum and the $\bar{q}b$ state. If this were the case, the process would factorize into $S \star T \star \Phi$. The factorization scale dependence of the soft factor S and of the collinear factor Φ would cancel separately with that of the hard-scattering kernel T, but the soft and collinear factors would be unrelated. The endpoint divergences prevent such a complete factorization. For our toy example we find instead a factorization formula that takes the schematic form

$$\langle\gamma|J|\bar{q}b\rangle = (C_0 + C_1) \star \phi_{\bar{q}b} + \Big[C_2 \star \phi_{\bar{q}bg}\Big]_\nu + \Big[\phi^\gamma_{q\bar{q}} \star C_3\Big]_\nu \star \phi_{\bar{q}b}. \tag{21}$$

The first term on the right-hand side represents a direct photon contribution; in the third term the partonic structure of the photon is resolved. The square brackets indicate the additional scale-dependence introduced by the endpoint divergences, which connect the second with the third term. If the scale ν is chosen such that the third term contains no large logarithm related to the endpoint divergence, we can interpret it as a endpoint-subtracted hard-scattering contribution to the $\bar{q}b \to \gamma$ transition. For our toy integral, (10,13) show that this corresponds to taking ν^2 of order $2p' \cdot l$. The corresponding endpoint logarithm then resides in the second term, which we may call the "soft overlap" contribution (since a soft line connects the initial state with the photon as seen from the middle column of Figure 3). The two terms are related via their ν-dependence, such that the sum is independent of the implementation of soft-collinear or "endpoint" factorization. A similar structure is expected for the $B \to \pi$ form factor [12].

3 Heavy-to-light transitions in SCET(c,s)

The effective theory representation of the heavy-to-light transition currents $\bar{\psi}\Gamma Q$ is obtained in two steps: first the hard modes are integrated out, and the current is described in soft-collinear effective theory including hard-collinear modes. We shall denote this theory by SCET(hc,c,s) (also called $\mathrm{SCET_I}$ in the literature [15]). This step, in which it is not necessary to distinguish hard-collinear and collinear, has already been discussed in

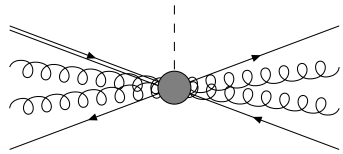

Figure 4: Kinematics of an exclusive heavy-to-light transition in SCET(c,s). The heavy quark and the soft partons in the B meson must be converted into a cluster of collinear partons.

[16, 17]. We will be mainly concerned with the second matching step, in which the hard-collinear modes are integrated out and the transition current is finally represented in terms of operators constructed only from soft and collinear fields. We refer to the theory of soft and collinear fields as SCET(c,s) (also called $\mathrm{SCET_{II}}$). The kinematics of a heavy-to-light transition is illustrated in Figure 4. In contrast to [17] the invariant mass of the final state is restricted to order $\lambda^4 \sim \Lambda^2$ as appropriate to an exclusive decay. This implies that the final state must now consist only of collinear lines, since the addition of a soft line would increase the virtuality to λ^2. The initial state is described by a heavy quark and soft lines with total invariant mass near m_b^2. The SCET(c,s) transition current has to turn a cluster of soft modes with the quantum numbers of the $\bar{B}$ meson into a cluster of collinear lines with the quantum numbers of the final-state meson.

We begin with a brief description of the SCET(c,s) fields, gauge symmetries and Lagrangian. This theory is in many ways simpler than SCET(hc,c,s), because the dynamics of the soft-collinear transition resides only in the effective current. We then discuss in detail the representation of the heavy-to-light current for an exclusive decay. In this section we restrict ourselves to tree-level matching. The general case will be considered in Section 4 to the extent that is necessary to prove the factorization of form factors. However, we briefly sketch the structure of transition operators and their coefficient functions beyond tree level at the end of this section.

3.1 Elements of SCET(c,s)

Fields. The SCET(c,s) Lagrangian and operators are built from a collinear light quark field ξ_c, a collinear gluon field A_c, and soft light quark, heavy quark and gluon fields, denoted by q_s, h_v, A_s, respectively. As in [17] we assume that collinear fields describe particles with large momentum in the direction of the light-like vector n_-. n_+ is another light-like vector, satisfying $n_-n_+ = 2$, and v will be the velocity vector labelling soft heavy quark fields. We will present our results in a general frame subject to the conditions $n_+v \sim 1$, $n_-v \sim 1$, $v_\perp \sim \lambda^2$.

The scaling of quark and gluon fields can be read off from the corresponding propagators

in momentum space. For the quark fields one finds

$$\xi_c = \frac{\not{n}_-\not{n}_+}{4}\,\psi_c \sim \lambda^2, \qquad h_v = \frac{1+\not{v}}{2}\,Q_v \sim \lambda^3, \qquad q_s \sim \lambda^3. \tag{22}$$

The "opposite" projections of the full collinear quark field ψ_c and the full soft heavy quark fields Q_v (defined as the heavy quark field with the rapid variations $e^{-im_b vx}$ removed) are λ^2 suppressed, and integrated out. Gluon fields scale as the corresponding derivatives,

$$n_+A_c \sim 1, \qquad A_{\perp c} \sim \lambda^2, \qquad n_-A_c \sim \lambda^4, \qquad A_s \sim \lambda^2. \tag{23}$$

In deriving this, we used that the integration measure $d^4x \sim 1/\lambda^8$, when the integral is over products of only collinear fields or products of only soft fields. This follows from the fact that an x-component scales inversely to the corresponding momentum component.

Multipole expansion. Since soft and collinear fields have significant variations over different length scales in the n_- and n_+ directions, they have to be multipole-expanded in products of soft and collinear fields. The multipole expansion in SCET(c,s) is different from the multipole expansion defined in [17], which applies to a theory with hard-collinear and soft fields, and no collinear fields. Here we need

$$\begin{aligned}
\phi_s(x) &= \phi_s(x_-) + \frac{n_-x}{2}\,[n_+\partial\,\phi_s]\,(x_-) + \ldots, \\
\phi_c(x) &= \phi_c(x_+) + \frac{n_+x}{2}\,[n_-\partial\,\phi_c]\,(x_+) + \ldots,
\end{aligned} \tag{24}$$

where

$$x_- \;\equiv\; n_+x\,\frac{n_-}{2} + x_\perp, \qquad x_+ \;\equiv\; n_-x\,\frac{n_+}{2} + x_\perp. \tag{25}$$

The correction terms in the two expansions of (24) are both λ^2 suppressed relative to the leading terms.

Gauge symmetry. The effective theory should be invariant under collinear and soft gauge transformations, defined as the restriction of gauge functions $U(x)$ to the corresponding spatial variations. The implementation of gauge transformations in the effective theory is not unique, since field redefinitions or applications of the field equations can be used to alter the gauge-transformation properties [17, 19, 27].

In SCET(c,s) collinear and soft fields decouple at leading power in the λ expansion as will be seen below. Furthermore, the product of a collinear and a soft field has hard-collinear momentum modes, therefore general soft gauge transformations acting on collinear fields (and vice versa) are not allowed. A natural choice is then to define [20]

$$\begin{aligned}
&\xi_c \;\to\; U_c\,\xi_c, \qquad A_c \;\to\; U_cA_cU_c^\dagger + \frac{i}{g}U_c\left[\partial, U_c^\dagger\right], \\
&h_v \;\to\; U_s\,h_v, \qquad q_s \;\to\; U_s\,q_s, \qquad A_s \;\to\; U_sA_sU_s^\dagger + \frac{i}{g}U_s\left[\partial, U_s^\dagger\right].
\end{aligned} \tag{26}$$

Lagrangian. The SCET(c,s) Lagrangian describes interactions of soft and collinear fields. In general, there can be scattering processes of the type $s + c \to s + c$. It has been shown in [20] for quark scattering that these interactions are power-suppressed. Below we extend this to gluons and derive the explicit form of the leading power-suppressed interactions. Because of the decoupling of soft and collinear modes at leading power, the Lagrangian is simply

$$\mathcal{L}^{(0)} = \mathcal{L}_s + \mathcal{L}_c \tag{27}$$

at leading power, with

$$\begin{aligned}
\mathcal{L}_c &= -\frac{1}{2}\,\mathrm{tr}\,(F_{\mu\nu c}F_c^{\mu\nu}) + \bar{\xi}_c \left(in_- D_c + (i\not{D}_{\perp c} - m)\,\frac{1}{in_+ D_c}\,(i\not{D}_{\perp c} + m)\right)\frac{\not{n}_+}{2}\,\xi_c, \\
\mathcal{L}_s &= -\frac{1}{2}\,\mathrm{tr}\,(F_{\mu\nu s}F_s^{\mu\nu}) + \bar{q}_s\,(i\not{D}_s - m)\,q_s + \mathcal{L}_{\mathrm{HQET}}
\end{aligned} \tag{28}$$

and $m \sim \lambda^2$ the light quark mass. The heavy quark interactions with soft fields are described by the standard heavy quark effective theory (HQET) Lagrangian $\mathcal{L}_{\mathrm{HQET}} = \bar{h}_v\, iv \cdot D_s\, h_v + \ldots$

For the kinematic situation shown in Figure 4 scattering processes $s + c \to s + c$ cannot occur. On the other hand, the crossed process $s + s \to c + c$ is not possible by momentum conservation in the $n_\mp$ directions, since $n_+ p > 0$ for a collinear momentum and $n_- l > 0$ for a soft momentum. It follows that insertions of the soft-collinear interaction terms from the sub-leading Lagrangian have zero $\langle \pi | \ldots | \bar{B} \rangle$ matrix elements, so that we can simply work with the Lagrangian without soft-collinear interactions to any accuracy. (See [25] for a related discussion.) Beyond leading power there are additional terms in the collinear Lagrangian $\mathcal{L}_c$, which arise upon integrating out heavy-quark loops with external collinear lines. Heavy-quark loops also generate additional soft interaction terms, which correct $\mathcal{L}_s$, which also includes the $1/m_b$ suppressed terms from the HQET Lagrangian. However, the soft-collinear interactions all reside in the effective current.

States and hadronic matrix elements. The B meson states in the effective theory may be defined as the eigenstates of the leading-order soft Hamiltonian. These states are identical to those used in HQET. Alternatively, if we regard the states as the eigenstates of the exact soft Hamiltonian, these states coincide with the B meson states of full QCD, since the soft Lagrangian to all orders is equivalent to the full QCD Lagrangian. Similarly, the light meson state in SCET(c,s) may be defined as the eigenstate of the leading-order collinear Hamiltonian. This Hamiltonian is equivalent to QCD without heavy quarks, so the pion state in the effective theory is the same as in QCD (without heavy quarks). Defining the pion with respect to the exact collinear Hamiltonian implies that the pion state is the same as in full QCD with heavy quarks. In the following we adopt the convention that the states are defined with respect to the exact soft or collinear Hamiltonians, so that we do not distinguish the effective theory states from those in QCD. It would be a simple matter to make the λ dependence of the states explicit. An implicit assumption here is

that the separation of collinear and soft modes is done without an explicit cut-off (dimensional or analytic regularization). With a regularization that breaks boost invariance the collinear Lagrangian is not equivalent to the full QCD Lagrangian. From the conventional normalization of hadronic states it follows that

$$|B(p)\rangle \sim \lambda^{-3}, \qquad |\pi(p')\rangle \sim \lambda^{-2}, \tag{29}$$

where the light meson is assumed to be energetic.

That a pion is made only of collinear partons can be understood by noting that a collinear pion state can be obtained from a pion at rest by a large Lorentz boost. In a pion at rest all partons have momenta of order Λ, so the boosted system contains only collinear modes. Adding a soft parton to the collinear modes produces a configuration of invariant mass $m_b\Lambda$, which cannot contribute to the pion pole. On the other hand, with the same line of reasoning, a B meson consists only of soft partons (and a heavy quark), since adding a collinear mode produces a configuration far away from the B meson pole.

An apparent consequence of the absence of soft-collinear interactions and the nature of the states is that an expression

$$C * \langle\pi|f(\phi_c)g(\phi_s)|\bar{B}\rangle \tag{30}$$

where $f(\phi_c)$ ($g(\phi_s)$) is a non-local product of collinear (soft) fields and the star denotes convolutions, factorizes into

$$\langle\pi|f(\phi_c)|0\rangle * C * \langle 0|g(\phi_s)|\bar{B}\rangle. \tag{31}$$

As shown in Section 2 this should be considered as formal, since the collinear and soft convolution integrals can be divergent.

Another consequence is that the QCD current matrix element $\langle\pi(p')|\bar{u}\,\Gamma\, b|B(p)\rangle$ simply matches to $\langle\pi(p')|J_{\rm eff}|B(p)\rangle$, so the problem reduces to obtaining the effective current. Already at this point we may note that the apparently leading term vanishes,

$$\langle\pi(p')|\bar{\xi}_c\,\Gamma\, h_v|\bar{B}(p)\rangle = 0, \tag{32}$$

because the quantum numbers of the product of collinear fields (here the single field $\bar{\xi}_c$) do not match those of a pion, and the quantum numbers of the soft fields (here only h_v) do not match those of the $\bar{B}$ meson. This can be formalized by saying that the effective Lagrangian is invariant under separate phase transformations of the collinear, soft and heavy quark fields, so we can assign "collinear quark number" to products of operators, with ξ fields carrying collinear quark charge $+1$, $\bar{\xi}$ having charge -1 and all other fundamental fields charge 0. We shall see later that the first non-zero matrix element is suppressed by three powers of λ. This is the origin of the well-known $1/m_b^{3/2}$ suppression of heavy-to-light form factors at large recoil [11].

This leads to the important observation [15] that power-suppressed currents in the effective theory become relevant to the $B\to\pi$ form factor at leading power. The derivation of these currents in SCET(c,s) will be worked out at tree level below, and in more generality

but less explicitly in Section 4. Note that in the intermediate SCET(hc,c,s), where only hard modes are integrated out, the power-counting for hadronic matrix elements is not explicit. In particular in [17] the pion state included soft-collinear interactions and the suppression of the matrix element of $\bar{\xi}_{hc}\,\Gamma\,h_v$ (with $\xi_{\rm hc}$ denoting the hard-collinear quark field) with these pion states was not determined (see the discussion at the end of Section 5.3 of [17] and in [20]).

Light-cone gauge and Wilson lines. It will sometimes be convenient – especially for the following tree-level matching of SCET(hc,c,s) to SCET(c,s) – to choose the gauge

$$n_+A_{hc} = n_+A_c = n_-A_s = 0. \tag{33}$$

The usefulness of $n_+A_{hc} = n_+A_c = 0$ gauge follows from the fact that SCET(hc,c,s) is non-local only due to the presence of Wilson lines in the direction of n_+, and due to the appearance of $(in_+\partial)^{-1}$. With $n_+A_{hc} = n_+A_c = 0$ all Wilson lines reduce to 1, and there are no fields of order 1. The usefulness of $n_-A_s = 0$ gauge is related to the fact that in SCET(hc,c,s) soft gluons decouple from collinear and hard-collinear gluons at leading order in λ in this gauge.

Once a particular result has been derived in this gauge, the collinear and soft gauge invariance is recovered by transforming the fields back to a general gauge using the gauge transformations (26). As explained in [19], the transformation matrices U_c and U_s that accomplish this are the light-like Wilson lines

$$\begin{aligned} U_c^\dagger(x) &= W_c(x) = P\,\exp\left(ig\int_{-\infty}^0 ds\,n_+A_c(x+sn_+)\right), \\ U_s(x) &= Y_s^\dagger(x) = P\,\exp\left(ig\int_0^\infty dt\,n_-A_s(x+tn_-)\right), \end{aligned} \tag{34}$$

and the corresponding gauge transformation of the fields can be written as

$$\begin{aligned} &\xi_c \;\to\; W_c^\dagger \xi_c, \qquad gA_c \;\to\; W_c^\dagger\,[iD_cW_c] \;\equiv\; \mathcal{A}_c, \\ &h_v \to Y_s^\dagger h_v, \qquad q_s \;\to\; Y_s^\dagger q_s, \qquad gA_s \;\to\; Y_s^\dagger\,[iD_sY_s] \;\equiv\; \mathcal{A}_s. \end{aligned} \tag{35}$$

(Here and in the following derivatives in square brackets act only on the expression to their right inside the bracket.) Because the Wilson lines transform as

$$Y_s \stackrel{U_c}{\to} Y_s, \qquad Y_s \stackrel{U_s}{\to} U_sY_s, \qquad W_c \stackrel{U_c}{\to} U_cW_c, \qquad W_c \stackrel{U_s}{\to} W_c, \tag{36}$$

the expressions on the right-hand side of (35) are gauge-singlets. The fields $\mathcal{A}_c$, $\mathcal{A}_s$ have been introduced in [20] as building blocks for manifestly gauge-invariant operators.

In a general gauge the Wilson lines emerge automatically from matching an infinite set of unsuppressed tree-level Feynman diagrams with attachments of n_+A_c to soft fields, and n_-A_s gluons to collinear fields as sketched in Figure 5. Indeed, at leading power these

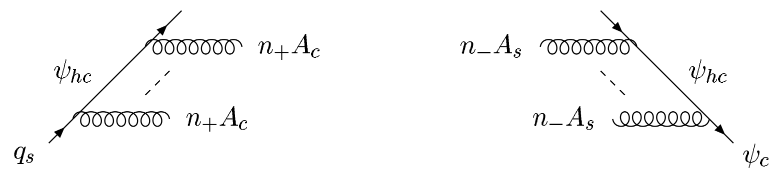

Figure 5: Infinite sets of Feynman graphs with attachments of soft gluons to collinear quarks and vice-versa. Integrating out the intermediate hard-collinear propagators leads to the Wilson lines $Y_s^\dagger$ and W_c, respectively, see (37).

diagrams result in[6]

$$\bar{\psi}_c \left(1 - g \not{A}_s \frac{1}{i\not{D} - m}\right) \simeq \bar{\xi}_c \left(1 + g n_- A_s \frac{1}{i n_- \overleftarrow{D}_s}\right) \simeq \bar{\xi}_c \, Y_s^\dagger,$$
$$\left(1 - \frac{1}{i\not{D} - m} \, g \not{A}_c\right) q_s \simeq \left(1 - \frac{1}{i n_+ D_c} \, g n_+ A_c\right) q_s \simeq W_c \, q_s. \tag{37}$$

In practice using the fixed gauge (33) in intermediate steps and restoring the gauge symmetry via (35) is more efficient, in particular when keeping track of Wilson lines arising from multi-gluon vertices in the non-Abelian theory.

3.2 Effective current at tree level

We now consider the representation of heavy-to-light transition currents $J = \bar{\psi}\Gamma Q$, where Γ is a Dirac matrix, in SCET(c,s). We are only concerned with tree diagrams in this subsection.

In the following we introduce different fields for hard-collinear and collinear modes. We work with the gauge $n_+A_c = n_+A_{hc} = n_-A_s = 0$ and present the gauge-invariant result only at the end. Integrating out hard intermediate heavy-quark propagators in tree diagrams, we obtain the current in SCET(hc,c,s) in the form

$$J(x) = e^{-i m_b v x} \left[\bar{\psi} \, \Gamma \, \mathcal{Q}\right](x), \tag{38}$$

where

$$\begin{aligned} \psi &= \xi_c + \eta_c + \xi_{hc} + \eta_{hc} + q_s \\ &= \xi_c + \xi_{hc} + q_s - \frac{1}{i n_+ D_s} \frac{\not{n}_+}{2} \left((i\not{D}_\perp + m) \left(\xi_c + \xi_{hc}\right) + \left(g \not{A}_{\perp c} + g \not{A}_{\perp hc}\right) q_s \right), \\ \mathcal{Q} &= \left(1 + \frac{i\not{D}_s}{2 m_b}\right) h_v - \frac{1}{n_- v} \frac{\not{n}_-}{2 m_b} \left(g \not{A}_{\perp c} + g \not{A}_{\perp hc}\right) h_v + O(\lambda^4 h_v) \end{aligned} \tag{39}$$

[6]We do not write out the $+i\epsilon$ prescription on the propagators, which reads $+i\epsilon$ for $1/(in_+\partial)$ and $-i\epsilon$ for $1/(in_-\partial)$. This follows, because the internal hard-collinear propagators are always space-like, with $(p' - l)^2 \simeq -n_- l \, n_+ p' < 0$, where $n_+ p' > 0$ describes an outgoing collinear momentum, and $n_- l > 0$ an incoming soft momentum. Hence in position space $1/(in_+\partial \, in_-\partial + i\epsilon)$ is $1/(in_+\partial + i\epsilon) \, 1/(in_-\partial - i\epsilon)$.

with η_c, η_{hc} the small components of the (hard-)collinear spinor field. This corresponds to the expressions given in [17, 19], apart from the fact that we have made explicit the presence of two different quark and gluon modes by substituting $\phi_c \to \phi_c + \phi_{hc}$ ($\phi = \xi, A$), and did not perform the multipole expansion.

Integrating out the hard-collinear fields at tree level expresses them in terms of soft and collinear fields as solutions to the classical field equations. In the following we integrate out the hard-collinear quark field first and obtain the effective field ψ depending of soft and collinear fields, and the hard-collinear gluon field. Inserting this into the gluon field equation determines the hard-collinear gluon field in terms of soft and collinear fields. Substituting back into the expression for ψ also determines the quark field. The SCET(c,s) representation of the current at tree level then follows by inserting the expressions for the hard-collinear fields into (39). We obtain the hard-collinear fields below, but anticipate here that $\xi_{hc} \sim \lambda^3$ and $A_{\perp hc} \sim \lambda^3$ for the classical fields. We used this to truncate the expansion of $\mathcal{Q}$ at order λ^6. None of the terms counted as λ^2 in [17] contribute to tree-level matching owing to the suppression of the classical hard-collinear fields. Also when $\bar{\psi}$ and $\mathcal{Q}$ are multiplied together only those terms with collinear quark number zero have non-zero $\langle \pi | \ldots | \bar{B} \rangle$ matrix elements.

3.2.1 Hard-collinear quark field

We first integrate out the hard-collinear quark field, which can be done exactly since the Lagrangian is bilinear in the quark field. This expresses the quark Lagrangian in terms of the soft and collinear fields and the hard-collinear gluon field. The latter is a function of the soft and collinear fields to be determined later from the gluon field equation.

The solution for the hard-collinear quark field can be derived in two ways, either by first integrating out the small component $\eta = \eta_c + \eta_{hc}$ of a collinear QCD spinor to go to SCET(hc,c,s) and then integrating out ξ_{hc} in SCET(hc,c,s), or by integrating out the hard-collinear field $\psi_{hc} = \xi_{hc} + \eta_{hc}$ in QCD and then integrating out η_c. We briefly comment on the two procedures.

In the first derivation, which is closer to the spirit of effective field theory, after eliminating η, we obtain the exact SCET Lagrangian before multipole expansion given as Eq. (6) of [19]. In collinear light-cone gauge $n_+ A_c = 0$, the quark Lagrangian reads

$$\begin{aligned}
\mathcal{L} \;=\; & \bar{\xi}\left(in_- D + [i\not{D}_\perp - m] \frac{1}{in_+ D_s} [i\not{D}_\perp + m] \right) \frac{\not{n}_+}{2} \xi + \bar{q}\,(i\not{D}_s - m)\, q \\
& - \bar{q}_s\, g\not{A}_{\perp c} \frac{\not{n}_+}{2} \frac{1}{in_+ D_s}\, g\not{A}_{\perp c} q_s + \bar{\xi}\, g\not{A}_{\perp c} q_s + \bar{\xi}\, \frac{\not{n}_+}{2} g n_- A_c q_s + \bar{\xi}\, \frac{\not{n}_+}{2}\, i\not{D}_\perp \frac{1}{in_+ D_s}\, g\not{A}_{\perp c} q_s \\
& + \text{h.c. of the } \bar{\xi}[\ldots] q_s \text{ terms.}
\end{aligned} \tag{40}$$

Then we substitute $\xi \to \xi_c + \xi_{hc}$ and $A_c \to A_c + A_{hc}$ in (40) and integrate out ξ_{hc}. Not all of the interactions vertices generated by this substitution can be realized due to the requirement of momentum conservation. For example, the term $\bar{\xi}\not{A}_{\perp c} q_s$ from (40) is

replaced by

$$(\bar{\xi}_c+\bar{\xi}_{hc})\,(\not{A}_{\perp c}+\not{A}_{\perp hc})q_s \simeq \bar{\xi}_c\,\not{A}_{\perp hc}q_s+\bar{\xi}_{hc}\,(\not{A}_{\perp c}+\not{A}_{\perp hc})q_s. \tag{41}$$

Although vertices with a single hard-collinear field cannot contribute to hard-collinear loops by momentum conservation, they cannot be simply omitted from the effective Lagrangian, because soft and collinear momenta add up to a hard-collinear momentum with non-generic transverse momentum of order λ^2. In fact, integrating out a hard-collinear field ϕ_{hc} at tree-level implies a solution $\phi_{hc}=f(\phi_c,\phi_s)$ of the classical field equations that can have only fluctuations with transverse momenta of order of the transverse fluctuations of soft and collinear fields.

In the second derivation directly from QCD we use the decomposition

$$\begin{aligned}(i\not{D}-m)^{-1} \;=\;& \frac{1}{in_+D+(i\not{D}_\perp+m)\,\frac{1}{in_-D}\,(i\not{D}_\perp-m)}\left(\frac{\not{n}_+}{2}+(i\not{D}_\perp+m)\,\frac{1}{in_-D}\,\frac{\not{n}_+\not{n}_-}{4}\right)\\ &+\frac{1}{in_-D+(i\not{D}_\perp+m)\,\frac{1}{in_+D}\,(i\not{D}_\perp-m)}\left(\frac{\not{n}_-}{2}+(i\not{D}_\perp+m)\,\frac{1}{in_+D}\,\frac{\not{n}_-\not{n}_+}{4}\right)\end{aligned} \tag{42}$$

of the inverse Dirac operator. We checked that the result for the hard-collinear quark field is the same in both methods. In both cases we found it useful to perform the calculation in light-cone gauge (33).

After eliminating the hard-collinear quark field as described, the result is expanded in powers of λ. In presenting the result of expansion we anticipate that the tree-level hard-collinear gluon field scales as $A_{\perp hc}\sim\lambda^3$ and $n_-A_{hc}\sim\lambda^4$ plus sub-leading terms. Below $A_{hc}^{(n)}$ denotes the term of order λ^n in the solution for the hard-collinear gluon field. Since the effective current requires the expression for ψ (see (39)), we present directly the result for

$$\psi=\xi_c+\eta_c+\xi_{hc}+\eta_{hc}+q_s\equiv\psi^{(2)}+\psi^{(3)}+\psi^{(4)}+\psi^{(5)}+\ldots \tag{43}$$

up to order λ^5, which we need later. The expansion begins with $\psi^{(2)}=\xi_c$, and the sub-leading terms read

$$\begin{aligned}\psi^{(3)} \;=\;& \left(1+\frac{1}{in_-\partial}\,g\not{A}_{\perp c}\,\frac{\not{n}_-}{2}\right)q_s,\\ \psi^{(4)} \;=\;& \frac{1}{in_+\partial}\,(i\not{D}_{\perp c}+g\not{A}_{\perp s}+m)\,\frac{\not{n}_+}{2}\,\xi_c\\ &-\frac{1}{in_-\partial}\left((i\not{D}_{\perp c}+g\not{A}_{\perp s}+m)\,\frac{1}{in_+\partial}\,g\not{A}_{\perp s}+g\not{A}_{\perp s}\,\frac{1}{in_+\partial}\,(i\not{D}_{\perp c}-m)\right)\xi_c\\ &-\frac{1}{in_-\partial}\,gn_-A_{hc}^{(4)}\,\xi_c+\frac{1}{in_-\partial}\,g\not{A}_{\perp hc}^{(3)}\,\frac{\not{n}_-}{2}\,q_s.\end{aligned} \tag{44}$$

The complete result for $\psi^{(5)}$ is rather involved. To simplify the presentation we neglect terms that have vanishing matrix elements in transitions to flavour-non-singlet mesons. In

this case we only need

$$\psi^{(5)} = \frac{1}{in_+\partial}\, g\not{A}^{(3)}_{\perp hc}\, \frac{\not{n}_+}{2}\, \xi_c - \frac{1}{in_-\partial}\bigg((i\not{D}_{\perp c} + g\not{A}_{\perp s} + m)\, \frac{1}{in_+\partial}\, g\not{A}^{(3)}_{\perp hc} + g\not{A}^{(3)}_{\perp hc}\, \frac{1}{in_+\partial}\, (i\not{D}_{\perp c} + g\not{A}_{\perp s} - m)\bigg)\, \xi_c - \frac{1}{in_-\partial}\, gn_-A^{(5)}_{hc}\, \xi_c + \ldots, \qquad (45)$$

and the ellipses stand for all the other terms. These results have simple interpretations in terms of trees of hard-collinear fields. For instance, the second structure in $\psi^{(3)}$ comes from the leading term in the solution for ξ_{hc} and describes $\xi_{hc} \to A_{\perp c} q_s$ at the end of a branch of a hard-collinear tree. Terms involving the hard-collinear gluon describe more complicated trees initiated by a quark field, where a hard-collinear gluon is emitted, which then splits into collinear and soft fields according to the solution for the gluon field given below. In a general gauge, arbitrarily complicated trees with external n_+A_c and n_-A_s fields exist at a given power of λ. The advantage of light-cone gauge is that every branching costs factors of λ. The structure of general trees can be recovered from gauge invariance.

We note that $\psi^{(2)}$ has collinear quark number $+1$, while $\psi^{(3)}$ has collinear quark number 0. The solutions for the hard-collinear field below show that $A^{(3)}_{\perp hc}$ and $n_-A^{(5)}_{hc}$ have collinear quark number ± 1, while $A^{(4)}_{\perp hc}$ and $n_-A^{(4)}_{hc}$ have collinear quark number 0. It follows that $\psi^{(n)}$ has odd (even) collinear quark number for even (odd) n. This will be useful later, when we discuss the matrix elements of the effective heavy-to-light transition current.

3.2.2 Hard-collinear gluon field

To complete the tree-level matching we consider the Lagrangian

$$-\frac{1}{4}\, F^2 + \mathcal{L}_{\rm quark} \qquad (46)$$

with $\mathcal{L}_{\rm quark}$ the SCET(hc,c,s) quark Lagrangian, in which the hard-collinear quark field is eliminated by its tree-level solution. We substitute $A = A_c + A_s + A_{hc}$ in the Yang-Mills Lagrangian and drop terms not allowed by momentum conservation as we did before for the quark Lagrangian. Integrating out the hard-collinear gluon field at tree level requires to solve the classical field equation. This is non-linear, but the hard-collinear gluon self-interactions turn out to be power-suppressed, so the field equation can be solved as an expansion in λ. With the ansatz $A_{\perp hc} = \sum_{n=2}^{\infty} A^{(n)}_{\perp hc}$, $n_-A_{hc} = \sum_{n=2}^{\infty} n_-A^{(n)}_{hc}$ (n denotes the power of λ), we find that the leading terms have $n = 3$ for $A_{\perp hc}$, and $n = 4$ for n_-A_{hc}.

The calculation gives

$$A^{(3)}_{\perp hc} = g\, T^A\, \frac{1}{(in_+\partial)(in_-\partial)}\, \Big\{\bar{q}_s \gamma_\perp T^A \xi_c + \text{h.c.}\Big\},$$

$$n_-A^{(4)}_{hc} = \frac{2g}{in_+\partial}\, \Big[A_{\mu_\perp c}, A^{\mu_\perp}_s\Big],$$

$$
\begin{aligned}
A_{hc}^{\mu_\perp (4)} &= \frac{g}{(in_+\partial)(in_-\partial)} \Big\{ i\mathcal{D}^{\mu_\perp}\Big[A_{\nu_\perp c}, A_s^{\nu_\perp}\Big] + i\mathcal{D}_{\nu_\perp}\Big(\Big[A_s^{\mu_\perp}, A_c^{\nu_\perp}\Big] + \Big[A_c^{\mu_\perp}, A_s^{\nu_\perp}\Big]\Big) \\
&\quad + \Big[in_+\partial A_c^{\mu_\perp}, \frac{2g}{in_+\partial}\Big[A_c^{\nu_\perp}, A_{\nu_\perp s}\Big]\Big] + \frac{in_+\partial}{2}\Big[A_s^{\mu_\perp}, n_- A_c\Big] + \frac{in_-\partial}{2}\Big[A_c^{\mu_\perp}, n_+ A_s\Big] \\
&\quad + \Big[iF_c^{\nu_\perp\mu_\perp}, A_{\nu_\perp s}\Big] + \Big[iF_s^{\nu_\perp\mu_\perp}, A_{\nu_\perp c}\Big] \\
&\quad + gT^A\,\bar{\xi}_c\Big(\not{A}_{\perp s}\frac{1}{in_+\partial}\gamma^{\mu_\perp}T^A + \gamma^{\mu_\perp}T^A\frac{1}{in_+\partial}\not{A}_{\perp s}\Big)\frac{\not{n}_+}{2}\,\xi_c \\
&\quad + gT^A\,\bar{q}_s\Big(\not{A}_{\perp c}\frac{1}{in_-\partial}\gamma^{\mu_\perp}T^A + \gamma^{\mu_\perp}T^A\frac{1}{in_-\partial}\not{A}_{\perp c}\Big)\frac{\not{n}_-}{2}\,q_s\Big\}, \\
n_- A_{hc}^{(5)} &= -\frac{2}{(in_+\partial)^2}\Big\{ i\mathcal{D}^{\mu_\perp}[in_+\partial A_{\mu_\perp hc}^{(3)}] - g\,\Big[in_+\partial A_c^{\mu_\perp}, A_{\mu_\perp hc}^{(3)}\Big] \\
&\quad + 2gT^A\Big\{\bar{\xi}_c T^A\Big(\frac{\not{n}_+}{2} - \frac{1}{in_-\partial}\,g\not{A}_{\perp c}\Big)q_s + \text{h.c.}\Big\}\Big\}. \qquad (47)
\end{aligned}
$$

Here $i\,\mathcal{D}^\mu\mathcal{O} = i\,\partial^\mu\mathcal{O} + g\,[A_c^\mu + A_s^\mu, \mathcal{O}]$ denotes the covariant derivative on a matrix-valued field. We see that at the lowest order the transverse hard-collinear gluon branches into a collinear and a soft quark, while $n_- A_{hc}$ splits into a collinear and a soft gluon. Substituting these expressions into (39) and (43,45) determines ψ and $\mathcal{Q}$ in terms of soft and collinear fields, and the effective current follows.

3.2.3 Soft-collinear interactions

As a by-product of the analysis we find the soft-collinear interactions in the SCET(c,s) Lagrangian including interactions suppressed with λ^3. We write

$$
\mathcal{L} = \mathcal{L}_s + \mathcal{L}_c + \mathcal{L}_{\text{s}-\text{c}}^{(2)} + \mathcal{L}_{\text{s}-\text{c}}^{(3)} + \ldots . \qquad (48)
$$

Soft and collinear fields are decoupled at leading power with $\mathcal{L}_s + \mathcal{L}_c$ given by (28). After using the gauge transformation (35) to convert all expressions into a manifestly gauge-invariant form, the tree-level interaction Lagrangian is

$$
\begin{aligned}
\mathcal{L}_{\text{s}-\text{c}}^{(2)} &= -\,\bar{\xi}_c W_c\,\not{\mathcal{A}}_{\perp s}\,\frac{1}{in_+\partial}\frac{\not{n}_+}{2}\,\not{\mathcal{A}}_{\perp s}\,W_c^\dagger\xi_c - \bar{q}_s Y_s\,\not{\mathcal{A}}_{\perp c}\,\frac{1}{in_-\partial}\frac{\not{n}_-}{2}\,\not{\mathcal{A}}_{\perp c}\,Y_s^\dagger q_s \\
&\quad + \frac{g^2}{2}\Big\{\bar{q}_s Y_s\,\gamma_{\perp\mu}\,T^A\,W_c^\dagger\xi_c + \text{h.c.}\Big\}\frac{1}{(in_+\partial)(in_-\partial)}\Big\{\bar{q}_s Y_s\,\gamma_\perp^\mu\,T^A\,W_c^\dagger\xi_c + \text{h.c.}\Big\} \\
&\quad + \frac{1}{g^2}\,\text{tr}\Big([\mathcal{A}_{\mu_\perp c}, \mathcal{A}_{\nu_\perp s}][\mathcal{A}_c^{\mu_\perp}, \mathcal{A}_s^{\nu_\perp}]\Big) + \frac{1}{g^2}\,\text{tr}\Big([\mathcal{A}_{\mu_\perp c}, \mathcal{A}_{\nu_\perp c}][\mathcal{A}_s^{\mu_\perp}, \mathcal{A}_s^{\nu_\perp}]\Big) \\
&\quad + \frac{1}{g^2}\,\text{tr}\Big([\mathcal{A}_{\mu_\perp c}, \mathcal{A}_{\nu_\perp s}][\mathcal{A}_s^{\mu_\perp}, \mathcal{A}_c^{\nu_\perp}]\Big) - \frac{1}{g^2}\,\text{tr}\Big([\mathcal{A}_{\mu_\perp c}, \mathcal{A}_s^{\mu_\perp}][\mathcal{A}_{\nu_\perp c}, \mathcal{A}_s^{\nu_\perp}]\Big). \qquad (49)
\end{aligned}
$$

In soft-collinear interactions $\int d^4x\,\mathcal{L}(\phi_c,\phi_s)$ we must count d^4x as $1/\lambda^6$, since $n_-x \sim 1$ is determined by the variation of collinear fields, $n_+x \sim 1/\lambda^2$ by the variations of soft fields, and $x_\perp \sim 1/\lambda^2$, so the vertices above give λ^2 suppression. $\mathcal{L}^{(3)}_{\rm s-c}$ is lengthy and will not be presented here. Recall that in products of soft and collinear fields all fields are multipole-expanded according to (24), i.e. soft fields are taken at x_- and collinear fields at x_+ in the above interaction terms. This implies that $1/(in_+\partial)$ commutes with soft fields, and $1/(in_-\partial)$ commutes with collinear fields. Note that in (49) the $i\epsilon$-prescription of the $1/(in_\mp\partial)$ and the integration path in the Wilson lines have to be adjusted according to whether one describes incoming or outgoing soft or collinear particles.

The Lagrangian is symmetric under interchange of soft and collinear fields together with $n_+ \leftrightarrow n_-$, because soft and collinear fields can be transformed into each other by a large Lorentz-boost [20]. The existence of the interactions in the first line has already been mentioned in [20]. The gluon self-interactions in the third and fourth line could be written more compactly using the Jacobi identity, but we prefer the above symmetric version.

We should emphasize that the soft-collinear interactions do not contribute to the matrix elements that define heavy-to-light form factors as explained above.

3.2.4 Effective current

We now return to the task of determining the SCET(c,s) heavy-to-light current at tree level. The current follows from inserting the expression for ψ from (43,45) and for the hard-collinear gluon field from (47) into (38,39). We write the result of this as

$$J(x) = e^{-im_b vx}\,\left[J^{(0)}_{\rm eff}(x) + J^{(1)}_{\rm eff}(x) + J^{(2)}_{\rm eff}(x) + J^{(3)}_{\rm eff}(x) + \ldots\right] \tag{50}$$

with the superscript indicating the suppression by powers of λ relative to the leading term $J^{(0)}_{\rm eff}$. Restoring the gauge symmetry the first two terms in the expansion read

$$\begin{aligned} J^{(0)}_{\rm eff} &= \bar\xi_c W_c\,\Gamma\, Y_s^\dagger h_v, \\ J^{(1)}_{\rm eff} &= \bar q_s Y_s\, \not{\!\!A}_{\perp c}\,\frac{1}{in_-\overleftarrow{\partial}}\,\frac{\not n_-}{2}\,\Gamma\, Y_s^\dagger h_v. \end{aligned} \tag{51}$$

For $\langle\pi|J^{(n)}_{\rm eff}|\bar B\rangle$ to be non-zero, the collinear fields of the operator product must have the quantum numbers of the pion, and the soft fields those of the $\bar B$ meson. Hence $J^{(0)}_{\rm eff}$ gives no contribution, as already discussed earlier. $J^{(1)}_{\rm eff}$ does have the correct collinear and soft quantum numbers, except the collinear fields (namely $\not{\!\!A}_{\perp c}$) are colour-octet, so the matrix element is again zero. The next term in the expansion is (in light-cone gauge)

$$J^{(2)}_{\rm eff} = \bar\psi^{(4)}\Gamma h_v - \frac{1}{n_-v}\,\bar\xi_c\,\Gamma\,\frac{\not n_-}{2m_b}\,g\not{\!\!A}_{\perp c}h_v, \tag{52}$$

which carries odd collinear quark number, implying zero matrix element.

This leaves

$$J^{(3)}_{\rm eff} = \bar\psi^{(5)}\Gamma h_v - \frac{1}{n_-v}\,\bar\psi^{(3)}\Gamma\frac{\not n_-}{2m_b}\,g\not{\!\!A}_{\perp c}h_v - \frac{1}{n_-v}\,\bar\xi_c\,\Gamma\,\frac{\not n_-}{2m_b}\,g\not{\!\!A}^{(3)}_{\perp hc}h_v, \tag{53}$$

as the leading effective current. Substituting $A^{(3)}_{\perp hc}$ from (47) we obtain the third term in the gauge-invariant representation

$$-\left(\frac{g^2}{(in_+\partial)(in_-\partial)}\,\bar{q}_s Y_s\,\gamma^{\mu_\perp}T^A W_c^\dagger \xi_c\right)\frac{1}{n_- v}\,\bar{\xi}_c W_c\,\Gamma\frac{\not{n}_-}{2m_b}\,\gamma_{\mu_\perp}T^A Y_s^\dagger h_v. \tag{54}$$

This can be identified with the tree diagram, where a transverse gluon is exchanged between the heavy quark and the spectator quark. The inverse differential operator is the hard-collinear gluon propagator in position space. With $\psi^{(3)}$ from (43) the second term on the right-hand side of (53) has collinear field content A_c or $A_c A_c$, which does not contribute to the matrix element, when the light meson is a flavour-non-singlet. The first term $\bar{\psi}^{(5)}\Gamma h_v$, which corresponds to trees rooted in the hard-collinear quark field at the current vertex, is by far the most complicated as seen from the expression for $\psi^{(5)}$ in (45). Although straightforward, it is not instructive to give the explicit, gauge-invariant form of this term. However, an important observation is that the field content of this term is not just $\bar{\xi}_c\xi_c\,\bar{q}_s h_v$, but also

$$\bar{\xi}_c A_{\perp c}\xi_c\,\bar{q}_s h_v \quad \text{and} \quad \bar{\xi}_c\xi_c\,\bar{q}_s A_{\perp s} h_v. \tag{55}$$

Eq. (45) shows that these terms corresponding to leading-power tree diagrams with external gluons all originate from $\bar{\xi}^{(5)}_{hc}\,\Gamma h_v$. (To see this, project (45) with $(\not{n}_-\not{n}_+)/4$.) On the other hand, the remaining projection

$$\begin{aligned}\bar{\eta}^{(5)}_{hc}\,\Gamma h_v &= \bar{\xi}_c\frac{\not{n}_+}{2}\,g\not{A}^{(3)}_{\perp hc}\left(in_+\overleftarrow{\partial}\right)^{-1}\Gamma h_v \\ &= \left(\frac{g^2}{(in_+\partial)(in_-\partial)}\,\bar{q}_s Y_s\gamma^{\mu_\perp}T^A W_c^\dagger\xi_c\right)\bar{\xi}_c W_c\frac{\not{n}_+}{2}\left(in_+\overleftarrow{\partial}\right)^{-1}\gamma_{\mu_\perp}T^A\Gamma\,Y_s^\dagger h_v\end{aligned} \tag{56}$$

involves only four-quark operators similar to (54). This observation will be crucial for the all-order factorization proof in Section 4, where we define the universal form factor as the matrix element of $\bar{\xi}^{(5)}_{hc}\,\Gamma h_v$ and show that the remaining terms involving four-quark operators factorize into a conventional hard-scattering term.

Summarizing, we conclude that the leading effective current is $J^{(3)}_{\rm eff} \sim \lambda^8$, so

$$\langle\pi(p')|J(0)|B(p)\rangle \approx \langle\pi(p')|J^{(3)}_{\rm eff}(0)|B(p)\rangle \sim \lambda^3 \sim m_b\left(\frac{\Lambda}{m_b}\right)^{3/2}, \tag{57}$$

where we restored the correct dimensions by inserting factors of m_b.[7] This reproduces the well-known heavy quark scaling of the heavy-to-light form factor at large recoil. However, here the scaling has been derived without recourse to any assumptions on the endpoint behaviour of the light-cone distribution amplitude. This result has also been obtained in [15] in the context of SCET(hc,c,s). However, since the matching of external lines with hard-collinear momentum to collinear lines that overlap with the meson state has not been considered there, the conclusion relied on the implicit assumption that the power counting

[7] Recall that for power counting we often set $m_b = 1$, so that λ counts factors of $\Lambda^{1/2}$.

of SCET(hc,c,s) can be applied. We may also note that the form factors have an expansion in $\lambda^2 \sim \Lambda_{\rm QCD}/m_b$ once the overall scaling is taken out. This has been suggested in [20] on grounds of the momentum scaling of soft and collinear fields. This does not completely cover the real situation, since the SCET(c,s) interactions and currents do have an expansion in λ (and not λ^2),

$$\mathcal{L}_{\rm s-c} \;=\; \mathcal{L}_{\rm s-c}^{(2)} + \mathcal{L}_{\rm s-c}^{(3)} + \ldots\;, \qquad J_{\rm eff} \;=\; J_{\rm eff}^{(0)} + J_{\rm eff}^{(1)} + \ldots\,. \tag{58}$$

However, the only source for odd powers of λ are the soft quark fields, q_s and h_v. Since collinear (and soft) quark number is conserved by the leading-power action, the combinations of $\mathcal{L}_{\rm s-c}$ and $J_{\rm eff}$ that can have non-zero matrix elements must always involve an even number of collinear quark or anti-quark fields. Therefore power-corrections to exclusive observables in SCET(c,s) are indeed given in terms of Λ/m_b.

3.2.5 Matrix element of the effective current

The matrix elements of the non-local operators in $J_{\rm eff}^{(3)}$ take a more familiar form in terms of convolutions with light-cone distribution amplitudes, when the factors $1/(in_\pm\partial)$ are replaced using the identities

$$\frac{1}{in_+\partial + i\epsilon}\phi(x) = -i\int_{-\infty}^0 ds\,\phi(x+sn_+), \qquad \frac{1}{in_-\partial - i\epsilon}\phi(x) = i\int_0^\infty dt\,\phi(x+tn_-). \tag{59}$$

Since all fields are multipole-expanded according to (24), the operator $1/(in_+\partial)$ commutes with soft fields since $[x+sn_+]_- = x_-$ (see (25) for the definition of $x_\pm$), and $1/(in_-\partial)$ commutes with collinear fields since $[x+tn_-]_+ = x_+$. Any operator can therefore be written as a convolution in variables s_i, t_j of a product of collinear fields with arguments $x+s_i n_+$ and a product of soft fields with arguments $x+t_j n_-$. We exemplify this for (54) and then discuss some important general features of the matrix element of $J_{\rm eff}^{(3)}$.

Applying (59) the matrix element of (54) turns into

$$\begin{aligned}
M \;\equiv\; & -g^2\int_{-\infty}^0 ds\int_0^\infty dt\,\langle\pi(p')|(\bar q_s Y_s)(tn_-)\,\gamma^{\mu_\perp}T^A(W_c^\dagger\xi_c)(sn_+) \\
& \qquad\times\frac{1}{n_-v}\,(\bar\xi_c W_c)(0)\,\Gamma\frac{\not n_-}{2m_b}\,\gamma_{\mu_\perp}T^A(Y_s^\dagger h_v)(0)|\bar B(p)\rangle \\
=\; & \frac{g^2C_F}{N_c}\,[\gamma^{\mu_\perp}]_{\beta\beta'}\,\frac{1}{n_-v}\,\Big[\Gamma\frac{\not n_-}{2m_b}\,\gamma_{\mu_\perp}\Big]_{\alpha\alpha'}\int_{-\infty}^0 ds\,\langle\pi(p')|(\bar\xi_{c\alpha}W_c)(0)\,(W_c^\dagger\xi_{c\beta'})(sn_+)|0\rangle \\
& \qquad\times\int_0^\infty dt\,\langle 0|(\bar q_{s\beta}Y_s)(tn_-)\,(Y_s^\dagger h_{v\alpha'})(0)|\bar B(p)\rangle,
\end{aligned} \tag{60}$$

where in the transition to the second line we performed the colour-singlet projection on the collinear and soft field products. The two matrix elements define the light-cone distribution amplitudes of the pion and the B meson,

$$\langle\pi(p')|(\bar\xi_{c\alpha}W_c)(sn_+)(W_c^\dagger\xi_{c\beta'})(0)|0\rangle = \frac{if_\pi}{4}\,n_+p'\left(\frac{\not n_-}{2}\gamma_5\right)_{\beta'\alpha}\int_0^1 du\,e^{iusn_+p'}\,\phi_\pi(u),$$

$$
\begin{aligned}
&\langle 0|(\bar{q}_{s\beta}Y)(tn_-)(Y^\dagger h_{v\alpha'})(0)|\bar{B}(p)\rangle \\
&\qquad = -\frac{if_B m_B}{4}\left(\frac{1+\not{v}}{2}\left[n_-v\,\not{n}_+\,\tilde{\phi}_{B+}(t)+n_+v\,\not{n}_-\,\tilde{\phi}_{B-}(t)\right]\gamma_5\right)_{\alpha'\beta}, \qquad (61)
\end{aligned}
$$

where $f_\pi = 131\,\mathrm{MeV}$ is the pion decay constant, f_B is the B decay constant and

$$
\tilde{\phi}_{B\pm}(t) \equiv \int_0^\infty d\omega\, e^{-i\omega t}\,\phi_{B\pm}(\omega). \qquad (62)
$$

This allows us to rewrite (60) as

$$
\begin{aligned}
M &= \frac{g^2 C_F}{N_c}\,\frac{f_B m_B}{4}\frac{f_\pi}{4}\,n_+p'\,\mathrm{tr}\left(\frac{1+\not{v}}{2}\,\not{n}_+\gamma_5\,\gamma^{\mu_\perp}\frac{\not{n}_-}{2}\gamma_5\Gamma\frac{\not{n}_-}{2m_b}\,\gamma_{\mu_\perp}\right) \\
&\qquad\times\int_0^1 du\,\phi_\pi(u)\int_{-\infty}^0 ds\,e^{i(1-u)sn_+p'}\int_0^\infty d\omega\,\phi_{B+}(\omega)\int_0^\infty dt\,e^{-i\omega t} \\
&= \frac{g^2 C_F}{N_c}\,\frac{f_B m_B}{4}\frac{f_\pi}{4}\,\mathrm{tr}\left(\frac{1-\not{v}}{2}\,\not{n}_+\not{n}_-\Gamma\frac{\not{n}_-}{2m_b}\right)\int_0^1 du\,\frac{\phi_\pi(u)}{1-u}\int_0^\infty\frac{d\omega}{\omega}\,\phi_{B+}(\omega), \qquad (63)
\end{aligned}
$$

which is the desired representation in terms of convolutions with light-cone distribution amplitudes.

With the standard assumption that $\phi_\pi(u)$ vanishes near the endpoints $u = 0, 1$ and that $\phi_{B+}(\omega) \to 0$ as $\omega \to 0$, the integrals converge in this particular case. But this is not true in general. The terms in $\bar{\psi}^{(5)}\Gamma h_v$ contain more powers of $1/(in_\pm\partial)$. An additional factor of $1/(in_-\partial)$ converts to a factor $1/\omega$ in the convolution, and a factor $1/(in_+\partial)$ results in $1/u$, $1/(1-u)$ or 1, depending on which fields it operates on. The convolution integrals may then be divergent. Furthermore we note that the terms (55) with the additional collinear or soft gluon fields result in contributions from three-particle light-cone distribution amplitudes in the pion or the B meson, respectively. The existence of endpoint divergences, which prevents the application of the standard hard-scattering formalism to the heavy-to-light form factors has been known before, but the contribution of three-particle amplitudes at leading power is a new and perhaps unexpected result. Interestingly, this also appears to happen in QCD sum rule calculations of the form factor [28]. In Figure 6 we show a sample of tree diagrams contained in $\bar{\psi}^{(5)}\Gamma h_v$, which are not power-suppressed.

3.2.6 Extension to photons

We briefly sketch the extension of our results to the $B \to \gamma$ transition form factors. These have been studied intensively recently [5, 6, 7, 8], since in $B \to \gamma l\nu$ decay theoretical issues related to the light-cone structure of the B meson and factorization in exclusive B decays can be studied without the interference of hadronic final state effects, at least in first approximation.

The electromagnetic coupling is taken into account by substituting

$$
gA_c \to gA_c^A T^A + e\mathcal{Q}A_{(\gamma)}, \qquad (64)
$$

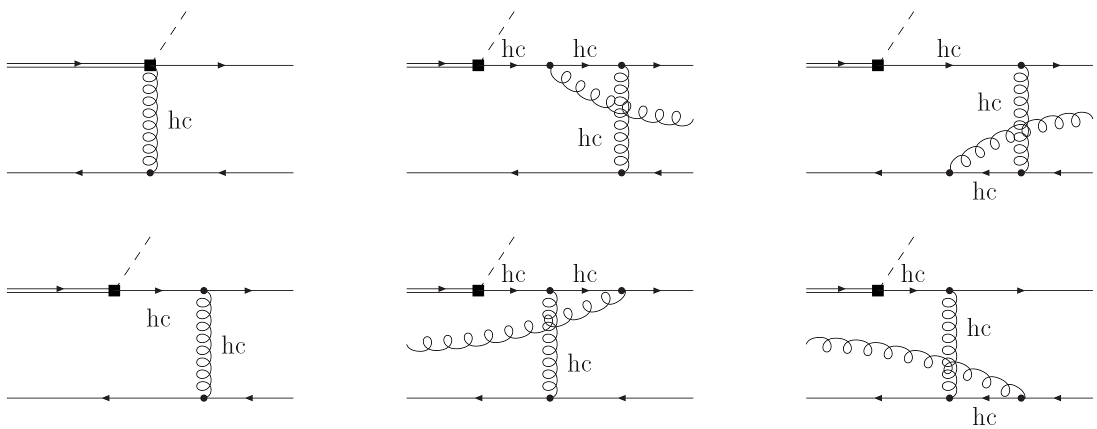

Figure 6: Tree diagrams that contribute to the form factor at leading power in light-cone gauge. Incoming lines are soft and outgoing lines are collinear.

in the expressions for the effective Lagrangian and heavy-to-light transition current. Here $A_{(\gamma)}$ denotes a collinear photon field, and $\mathcal{Q}$ the electric charge operator. In the following we always consider the physical transverse polarizations of the photon so that $n_\pm A_{(\gamma)} = 0$, and expand to first order in the electromagnetic coupling constant.

The leading order effective Lagrangian now contains the standard electromagnetic coupling $e\bar{\xi}_c \mathcal{Q} \not{A}_{\perp(\gamma)} \xi_c$ to collinear quarks. This allows unsuppressed interactions of photons with any number of collinear quark and gluons fields. As a consequence matrix elements $\langle \gamma(p')|\mathcal{O}|\bar{B}(p)\rangle$ are non-zero even if $\mathcal{O}$ contains no photon field, making the photon behave similar to a vector meson. Such terms are usually referred to as related to the "hadronic structure of the photon". Hence all the terms at order λ^3 in the analysis of transitions to mesons can be taken over to the $\bar{B} \to \gamma$ transition.

However, there exist additional terms involving the collinear photon field. Such terms are referred to as "direct photon" contributions. The corresponding operators $\mathcal{O}$ contain the photon field and soft fields, but no collinear quark and gluon fields. The most important modification of the analysis for mesons arises due to the presence of an additional term

$$J^{(1)}_{\mathrm{eff},(\gamma)} = ee_q\, \bar{q}_s Y_s\, \not{A}_{\perp(\gamma)}\, \frac{1}{in_- \overleftarrow{\partial}}\, \frac{\not{n}_-}{2}\, \Gamma\, Y_s^\dagger h_v = ee_q\, \bar{q}_s\, \not{A}_{\perp(\gamma)}\, \frac{1}{in_- \overleftarrow{D}_s}\, \frac{\not{n}_-}{2}\, \Gamma\, h_v \qquad (65)$$

in the first order current, which has a non-vanishing $\langle \gamma| \ldots |\bar{B}\rangle$ matrix element. We therefore conclude that

$$\langle \gamma(p')|J(0)|\bar{B}(p)\rangle \sim \lambda \sim m_b \left(\frac{\Lambda}{m_b}\right)^{1/2}, \qquad (66)$$

i.e. the direct photon contribution is a factor of $m_b/\Lambda_{\mathrm{QCD}}$ larger than the hadronic contribution. The operator $J^{(1)}_{\mathrm{eff},(\gamma)}$ is the tree-level equivalent of the operator written down in [8]. Allowing for a non-trivial coefficient function of the operator due to radiative corrections, and writing the operator in convolution form as discussed above, we reproduce the

factorization theorem for $B \to \gamma l\nu$ decay at leading order in the $1/m_b$ expansion in the form given in [8].

There exist further direct photon contributions, but they are all suppressed by at least two powers of λ, hence they are of the same order as the hadronic terms. For instance, photon radiation off the heavy quark field is obtained from substituting (64) into the effective heavy quark field $\mathcal{Q}$ in (39), which generates the additional term

$$-ee_b \, \frac{1}{n_- v} \, \bar{q}_s \, \Gamma \, \frac{\not{n}_-}{2m_b} \, \not{A}_{\perp(\gamma)} \, h_v \tag{67}$$

in $J_{\rm eff}^{(3)}$. Hence the direct photon coupling to the heavy quark also constitutes a $1/m_b$ correction to the leading transition form factor.

Giving the complete set of $1/m_b$ corrections does not provide further insight into the structure of the form factor. In general, the issues related to factorization for the $B \to \pi$ form factor at leading power, such as endpoint divergences and unsuppressed contributions from multi-particle distribution amplitudes, are all relevant to the $1/m_b$ correction for the $B \to \gamma$ form factor. However, this also implies that within the soft-collinear effective theory framework it is in principle possible to establish a factorization formula for the $B \to \gamma$ form factor, which includes power corrections.

3.3 Hard-collinear loops

Tree-level matching revealed that SCET(c,s) operators are non-local in two directions with factors in the product of collinear (soft) fields displaced in the n_+ (n_-) direction. The effective theory is local in the transverse coordinates. In this section we show that this structure is preserved beyond tree level.

It is simpler to begin the discussion in momentum space. Matching SCET(c,s) to SCET(hc,c,s) beyond tree level means that we consider an N-loop diagram in SCET(hc,c,s), where all loop momenta k_i are hard-collinear. The external momenta consist of a set of collinear momenta p_j, a set of soft momenta l_i, and, in the case of the heavy-to-light current, the heavy quark residual momentum r. The matching calculation is performed by Taylor-expanding the loop integrand under the assumption of hard-collinear loop momentum. Apart from numerator factors, which are polynomials in the momenta, the integrand can only contain massless propagators and factors related to the non-locality of SCET(hc,c,s) vertices. The massless propagators are expanded as

$$\frac{1}{(K+L+P)^2} = \frac{1}{K^2 + n_+ K n_- L + n_- K n_+ P + n_+ P n_- L} + \ldots, \tag{68}$$

where K (L, P) denotes a linear combination of hard-collinear (soft, collinear) momenta. In case of the heavy-to-light current, we exploit that the heavy quark line ends at the current vertex, so that the residual momentum can be routed such that it never enters a massless propagator. The non-locality of SCET(hc,c,s) vertices may supply factors of $1/(n_+(K+P))$. The only external vectors that enter the loop integrand denominator

are therefore $(n_+p_j)\, n_-/2$ and $(n_-l)\, n_+/2$, from which we can build the Lorentz-invariants $Q_{ij} = (n_-l_i)(n_+p_j)$. When the loop integrals are computed in dimensional regularization, the dimensionless short-distance part can only be of the form

$$f(Q_{i'j'}/Q_{ij}, \ln(Q_{ij}/\mu^2)) \tag{69}$$

that is an arbitrary function of ratios of invariants and a polynomial of logarithms of the factorization scale.

The dependence of a matching coefficient on an external momentum implies that the corresponding operator is non-local. Here the non-locality is due to the fact that the n_+ component of the hard-collinear momentum is not larger than the n_+ component of the external collinear momentum, and the n_- component of the hard-collinear momentum is not larger than the n_- component of the soft momentum. In position space the invariants Q_{ij} correspond to factors $1/(in_-\partial\, in_+\partial)$ acting on certain products of fields. It is more convenient to use (59) to represent these factors as integrations. The general form of a soft-collinear operator including its coefficient function reads

$$\begin{aligned} \int_{-\infty}^{\infty} ds_1 \ldots ds_n \int_{-\infty}^{\infty} dt_1 \ldots dt_m \, s_i^a t_j^b \, C\Big((s_{i'}t_{j'})/(s_i t_j), \ln(s_i t_j \mu^2)\Big) \\ \times [\phi_{s1}(x+t_1 n_-) \ldots \phi_{sm}(x+t_m n_-)]\, [\phi_{c1}(x+s_1 n_+) \ldots \phi_{cn}(x+s_n n_+)]. \end{aligned} \tag{70}$$

We extended the integration limits to infinity, since the precise limits depend on the details of how the multiple inverse derivative operators act on products of fields. The correct limits are then implemented through theta-functions in C. The two square brackets contain the non-local products of soft and collinear fields, respectively, which may contain derivatives or Dirac matrices depending on the specific case. The coefficient function C is dimensionless by construction. It should be noted that the s_i and t_j have mass dimension -1 and transform as $s_i \to \alpha s_i$, $t_j \to \alpha^{-1} t_j$ under the boost transformation $n_- \to \alpha n_-$, $n_+ \to \alpha^{-1} n_+$, under which the effective theory must remain invariant. Knowing the boost transformation of the SCET(hc,c,s) operator implies constraints on the possible occurrence of "free" factors of s_i and t_j, indicated above as $s_i^a t_j^b$. The Fourier transforms of the coefficient functions defined here appear as hard(-collinear) scattering kernels in factorization theorems. In general, all interaction and operator vertices in SCET(c,s) take the form of convolution kernels.

4 Factorization of heavy-to-light form factors

We now turn to the proof of the factorization formula (1) for B meson transition form factors to an energetic light meson. The light meson is assumed to be a flavour non-singlet pseudoscalar, such as the pion, but a similar result holds for vector mesons as discussed below.

The three independent form factors for $\bar{B}$ decays into a pion are defined by the following

Lorentz decompositions of bilinear quark current matrix elements:

$$\langle\pi(p')|\bar{u}\,\gamma^\mu b|\bar{B}(p)\rangle = f_+(q^2)\left[p^\mu + p'^\mu - \frac{m_B^2 - m_\pi^2}{q^2}\,q^\mu\right] + f_0(q^2)\,\frac{m_B^2 - m_\pi^2}{q^2}\,q^\mu, \quad (71)$$

$$\langle\pi(p')|\bar{u}\,\sigma^{\mu\nu}q_\nu b|\bar{B}(p)\rangle = \frac{i f_T(q^2)}{m_B + m_\pi}\left[q^2(p^\mu + p'^\mu) - (m_B^2 - m_\pi^2)\,q^\mu\right], \quad (72)$$

where m_B is the B meson mass, m_π the pion mass, and $q = p - p'$. The result to be shown is that for $i = +, 0, T$, and for $q^2 \ll m_B^2$ (such that the energy of the pion $E \gg \Lambda$),

$$f_i(q^2) = C_i(q^2)\,\xi_\pi(q^2) + \int_0^\infty d\omega \int_0^1 du\, T_i(q^2;\omega,u)\,\phi_{B+}(\omega)\,\phi_\pi(u), \quad (73)$$

up to corrections of order Λ/m_B, and that the following holds:

1. The coefficients C_i and T_i are dominated by short-distance physics at the scales m_B and $\sqrt{m_B\Lambda}$ and have expansions in α_s.

2. The form factor ξ_π is universal, i.e. independent of $i = +, 0, T$. (ϕ_{B+} and ϕ_π are light-cone distribution amplitudes defined later.)

3. The integrations over ω and u converge. In particular, there are no “endpoint singularities”.

The proof proceeds in three steps: we first determine the SCET(hc,c,s) operators, which give leading-power contributions to the form factors after matching them to SCET(c,s). We then define the universal form factor ξ_π. Finally, we show that the remainder $f_i - C_i\xi_\pi$ matches only on a particular type of SCET(c,s) operator, which factorizes into light-cone distribution amplitudes, and for which the convolution integrals are convergent. The strategy which we apply here is to avoid having to deal with endpoint divergences by applying a definition of $\xi_\pi(q^2)$ that subsumes all these effects. This comes at the price of not factorizing $\xi_\pi(q^2)$ into its hard-collinear, collinear and soft subprocesses. However, this is quite sufficient, since the main outcome of the formula (73) that the three independent form factors reduce to one plus a hard-scattering term does not require the factorization of $\xi_\pi(q^2)$.

4.1 Matching on SCET(hc,c,s): Integrating out hard modes

At leading order the flavour-changing QCD currents $\bar{\psi}\Gamma_i Q$ match to operators

$$\bar{\xi}_C W_C \Gamma'_j h_v \quad (74)$$

in SCET(hc,c,s), where Γ'_j is a basis of Dirac matrices. To avoid confusion with the label “c” for collinear fields in SCET(c,s) we use the label “C” to denote collinear quantities in SCET(hc,c,s), when they describe hard-collinear *and* collinear modes. The operators (74) scale with λ^4. The QCD operators have been matched to SCET at tree level up to order

λ^6 [17], and the complete operator basis is known at order λ^5 [29]. The λ scaling here is derived from counting the C-fields as hard-collinear, since this gives the smallest power, and covers the general case.

The tree-level matching calculation showed that the leading SCET(c,s) operators that have non-vanishing $\langle\pi|\ldots|\bar{B}\rangle$ matrix elements scale with λ^8. (The form factor then scales as $\lambda^3 \sim 1/m_b^{3/2}$ as expected.) However, it is not necessary to enumerate the complete basis of current operators in SCET(hc,c,s) up to order λ^8, since most of these operators contribute to the $\langle\pi|\ldots|\bar{B}\rangle$ matrix element only through time-ordered products with power-suppressed SCET(hc,c,s) interactions. Rather than giving the general form of the SCET(hc,c,s) current after integrating out the hard modes, we restrict our attention to those operators that contribute to the leading-power form factor. To identify these operators, we distinguish hard-collinear and collinear fields in the following.

For SCET(hc,c,s) operators that scale as λ^n, the power n is never smaller than the scaling of the product of fields involved, because there are no $1/(in_-\partial)$ factors in SCET(hc,c,s). We therefore first list the operators by their field content, using the gauge $n_+A_c = n_+A_{hc} = n_-A_s = 0$. The gauge-invariant representation of the operators will be constructed later.

Consider first operators without hard-collinear fields. To represent the quantum numbers of the $\bar{B}$ meson, the fields $\bar{q}_s h_v$ are needed. Adding collinear fields for the outgoing light meson, the only operator at order λ^8 is $\bar{q}_s h_v A_c$, but this cannot contribute to the $\langle\pi|\ldots|\bar{B}\rangle$ matrix element, since the collinear fields are colour-octet flavour-singlet. The leading purely soft-collinear operators with the correct quantum numbers arise at order λ^{10}, the two possibilities reading $\bar{q}_s h_v \bar{\xi}_c \xi_c$, $\bar{q}_s h_v A_c A_c$. The second operator is flavour-singlet and irrelevant for pions.[8] The first is generated by the hard contribution to the box graph representing gluon exchange from the heavy quark and the light quark to the spectator quark. As expected this results in a $1/m_b$ power correction to the form factor.

Any relevant SCET(hc,c,s) operator must therefore contain at least one hard-collinear field. With the power counting for hard-collinear fields, $\xi_{hc} \sim \lambda$, $A_{\perp hc} \sim \lambda$, $n_-A_{hc} \sim \lambda^2$, we obtain

$$\begin{array}{ll}
\lambda^4 & \bar{\xi}_{hc} h_v \\
\lambda^5 & \bar{\xi}_{hc} A_{\perp hc} h_v \\
\lambda^6 & 1)\ \bar{\xi}_c A_{\perp hc} h_v, \quad 2)\ \bar{\xi}_{hc} A_{\perp c} h_v, \quad 3)\ \bar{\xi}_{hc} A_{\perp s} h_v, \quad 4)\ \bar{\xi}_{hc} n_- A_{\rm hc} h_v, \\
 & 5)\ \bar{\xi}_{hc} A_{\perp hc} A_{\perp hc} h_v,
\end{array} \tag{75}$$

and so on. We then need to determine the λ suppression factors incurred when the hard-collinear fields convert into soft and collinear fields through time-ordered products. We show now that this implies that no operators with hard-collinear fields that scale as λ^7 or λ^8 in SCET(hc,c,s) contribute to the leading-power form factors, and that only a few of the operators listed above are actually relevant.

To find the suppression factors we inspect the interaction vertices of the SCET(hc,c,s) Lagrangian. There exist unsuppressed interactions among the hard-collinear modes but any

[8] To avoid heavy notation we do not indicate the flavour of the collinear and soft quark fields, which should be clear from the context.

interaction that couples hard-collinear to collinear and/or soft modes costs at least a factor of λ (in the light-cone gauges adopted above). It follows that SCET(hc,c,s) operators with hard-collinear fields at order λ^8 contribute to the form factors only at sub-leading order. To proceed we note that the SCET(c,s) current operators must have the field content

$$(\bar{q}_s h_v[\ldots])\ (\bar{\xi}_c \xi_c[\ldots]) \tag{76}$$

to match the quantum numbers of the $\bar{B}$ meson and the pion, where the first ellipses denote additional soft fields (gluons or quark-antiquark pairs), the second ellipses additional collinear fields, and $\bar{\xi}_c\xi_c$ is flavour non-singlet. In particular, the simplest operators are four-quark operators.[9] Suppose now that J_7 is a SCET(hc,c,s) current operator, which scales as λ^7, and contains at least one hard-collinear field. For this operator to contribute to the form factor at leading power, there should be a time-ordered product

$$\int d^4x\, T(J_7, \mathcal{L}_{\rm int}(x)) \tag{77}$$

of order λ^8 that matches onto the required four-quark operator. Besides h_v the current J_7 can contain only $\bar{q}_s$ or $\bar{\xi}_c$ (but not both) and no further soft or collinear fields, so the question is whether there is a SCET(hc,c,s) interaction that produces $\bar{q}_s\xi_c$ or $\bar{\xi}_c\xi_c$ at the price of only a single factor of λ. According to (40) the only possible candidate is

$$\mathcal{L}_{{\rm int},\xi q} = \bar{q}_s g \not{A}_{\perp hc}\xi_c, \tag{78}$$

but since this interaction contains a only single hard-collinear field, it must occur at the end of a branch of a hard-collinear tree. Tree-level matching tells us that the hard-collinear gluon that connects to this branch counts as λ^3 rather than λ, so the time-ordered product (77) is at least of order λ^9. More explicitly,

$$A_{\perp hc} \int d^4x\, \bar{q}_s g \not{A}_{\perp hc}\xi_c \sim \lambda^3, \tag{79}$$

since the contraction integrated over space-time scales as $1/\lambda^2$, as the hard-collinear propagator in momentum space. Since $A_{\perp hc}$ counted as λ when we determined the scaling of J_7, we see that the conversion of $A_{\perp hc}$ into $\bar{q}_s\xi_c$ costs a factor of λ^2. It follows that no operator that scales with λ^7 in SCET(hc,c,s) is relevant to the form factors at leading power.

We are left with the list (75), which we now reduce. Consider first the leading operator $J_4 = \bar{\xi}_{hc}h_v$. A non-zero matrix element requires at least two suppressed interaction vertices, since no single interaction can produce the required $\bar{\xi}_c\xi_c\bar{q}_s$ fields. The leading candidate is

$$\int d^4x\, d^4y\, T(J_4, \mathcal{L}_{{\rm int},\xi\xi}(x), \mathcal{L}_{{\rm int},\xi q}(y)) \tag{80}$$

with $\mathcal{L}_{{\rm int},\xi q}$ from (78) and $\mathcal{L}_{{\rm int},\xi\xi}$ an interaction of the form $\bar{\xi}_c[\ldots]\xi_{hc}$ from the collinear Lagrangian. The second interaction implies a factor of λ^2 as shown above, but it appears

[9] In case of flavour-singlet mesons the collinear fields may be gluons only, but we do not consider this case here.

that there are order λ interactions such as $\bar{\xi}_c A_{\perp,hc}(in_+\partial)^{-1}\partial_\perp \xi_{\rm hc}$ in $\mathcal{L}_{\text{int},\xi\xi}$, in which case the time-ordered product scales with λ^7. However, an additional factor of λ arises, because one can show that in an arbitrary diagram contributing to the time-ordered product, there is always an odd number of transverse derivatives. Although these scale as λ in general, at least one of them must act on an external field (in which case it scales as λ^2), since the integrations over an odd number of factors of internal transverse momenta vanish. Hence J_4 contributes to the relevant λ^8 current operators in SCET(c,s). Some of the tree-level terms discussed earlier can in fact be traced to this time-ordered product.

The time-ordered products needed to generate a non-vanishing $\langle \pi | \ldots | \bar{B} \rangle$ matrix element of the operator $J_5 = \bar{\xi}_{hc} A_{\perp hc} h_v$ are the same as for J_4. The additional λ suppression is not present here, because the operator J_5 already supplies one transverse gluon field, so J_5 also belongs to the set of operators relevant to the form factors at leading power. However, there is no tree-level contribution from J_5, since J_5 can contribute at order λ^8 only through hard-collinear loops.

Turning to the five operators that scale with λ^6 in the list (75), we can immediately discard all except the first one. For the operators 2) and 3) this follows, because we require the same time-ordered products as in (80) to obtain a non-vanishing matrix element, but the additional collinear or soft gluon field in the operator gives an extra λ^2 suppression. Similarly, the operators 4) and 5) can be dropped by comparing them to J_5. This leaves $J_6 = \bar{\xi}_c A_{\perp hc} h_v$, which gives a relevant λ^8 term through

$$\int d^4x \, T(J_6, \mathcal{L}_{\text{int},q\xi}(x)). \tag{81}$$

This contribution is already present at tree level (third term in (53)).

Having determined the field structure of the operators, we note that there are only three independent Dirac structures between $\bar{\xi}_c$ (or $\bar{\xi}_{hc}$) and h_v, which we choose as

$$\Gamma'_j = \{1, \gamma_5, \gamma_\perp^\mu\}. \tag{82}$$

The only SCET(hc,c,s) currents that we need are therefore of the form

$$\bar{\xi}_{hc}\Gamma'_j h_v, \quad \bar{\xi}_{hc} A^\mu_{\perp hc}\Gamma'_j h_v, \quad \bar{\xi}_c A^\mu_{\perp hc}\Gamma'_j h_v. \tag{83}$$

Any other operator that may appear in the representation of the QCD weak currents in SCET(hc,c,s) contributes only power corrections to the form factors. In the notation of [17] the three relevant operators descend from $J^{(A0)}$, $J^{(A1)}$ and $J^{(B1)}$.

To put the position arguments of the fields and gauge invariance in place, we note that fields are multipole-expanded in the position space formulation of SCET [17], and that the (hard-)collinear fields are in general separated in the n_+ direction along the light-cone due to the action of $(in_+\partial)^{-1}$. For the following gauge-invariant representation it is convenient to temporarily return to the formulation of SCET(hc,c,s), where a collinear field represents hard-collinear and collinear modes. The multipole expansion of soft fields around $x_- = (n_+x)n_-/2$ performed in [17, 19] is not appropriate now, because it applies to a theory with only hard-collinear and soft modes, where the transverse variations of soft

fields are always small compared to those of the hard-collinear fields. Gauge invariance in SCET(hc,c,s) in situations with collinear and hard-collinear modes has not been discussed in the literature so far in either of the two formulations of SCET (position space, hybrid), and we do not intend to give a systematic discussion here. In position space a simple remedy appears to be to include the transverse position in the definition of x_- and to multipole-expand the soft fields around $x_- = (n_+x)n_-/2 + x_\perp$. The gauge transformation of the transverse collinear field in Eq. (10) of [19] should then be replaced by

$$A_{\perp C} \to U_C\, A_{\perp C}\, U_C^\dagger + \frac{i}{g}\, U_C \left[D_{\perp \mathrm{s}}(x_-), U_C^\dagger\right]. \tag{84}$$

(Remember that the subscript "C" includes hard-collinear and collinear modes.) Contrary to the construction in [19] the gauge transformations do not obey homogeneous power counting rules in λ. This is not surprising, since the SCET(hc,c,s) Lagrangian has no homogeneous power counting, when hard-collinear and collinear modes are represented by a single field. The gauge-invariant form of the operators (83) is now given by

$$O_j^0(s) \;\equiv\; (\bar\xi_C W_C)(x+sn_+)\Gamma_j'\, h_v(x_-) \equiv (\bar\xi_C W_C)_s \Gamma_j'\, h_v \tag{85}$$

$$O_j^{1\mu}(s_1,s_2) \;\equiv\; (\bar\xi_C W_C)_{s_1} (W_C^\dagger i D_\perp^\mu W_C)_{s_2} \Gamma_j'\, h_v \tag{86}$$

with $D_\perp$ acting on everything to its right.[10] In light-cone gauge $W_C = 1$, and the second operator reduces to $\bar\xi_C A^\mu_{\perp hc}\Gamma_j' h_v$ and further λ suppressed terms, which we can drop.

The operators are multiplied by coefficient functions, whose tree-level values are corrected by hard loops. Lorentz invariance implies that the coefficient functions can depend only on the invariants m_b^2 and $m_b n_- v\, n_+ p_k'$ (both of order m_b^2) with p_k' referring to a set of independent external collinear or hard-collinear momenta. The other possible invariants are small compared to m_b^2 and Taylor-expanded before the computation of the loop integral [17]. Invariance under boosts, $n_- \to \alpha n_-$, $n_+ \to \alpha^{-1} n_+$, implies the combination $n_- v\, n_+ p_k'$, as can also be verified explicitly from the structure of loop integral denominators. The dependence of the coefficient function on an external momentum component results in non-locality and convolutions in coordinate space. If the position argument of the field is $x + s_i n_+$, we express the coefficient function in terms of the dimensionless and boost-invariant convolution variable $\hat s_i \equiv s_i m_b / n_- v$.

The QCD currents $\bar\psi \Gamma_i Q$ are now represented in SCET(hc,c,s) as

$$(\bar\psi \Gamma_i Q)(x) \;=\; e^{-im_b v\cdot x} \bigg\{ \sum_j \int d\hat s\, \tilde C_{ij}^{\,0}\,(\hat s, \frac{m_b}{\mu})\, O_j^0(s) + \frac{1}{m_b} \sum_j \int d\hat s_1 d\hat s_2\, \tilde C_{ij}^{1\mu}(\hat s_1, \hat s_2, \frac{m_b}{\mu})\, O_j^{1\mu}(s_1,s_2) \bigg\} + \ldots, \tag{87}$$

where further terms that do not contribute to the leading-power operators after matching to SCET(c,s) are not written explicitly. This is the main result of the first step in

[10] The interpretation of the subscript "s" as denoting a soft field or the position argument $x + sn_+$ should be clear from the context.

the factorization proof. The factor $1/m_b$ in the second line has been inserted so that the coefficient functions are dimensionless. The indices 'ij' are schematic and contain Lorentz indices in general. The coefficient functions are boost-invariant Lorentz tensors constructed from $n_\mp^\mu$, $g^{\mu\nu}$, $\epsilon^{\mu\nu\rho\sigma}$ and n_-v.[11] For instance, if $\Gamma_i = i\sigma^{\mu\nu}$ and $\Gamma'_j = \gamma_\perp^\rho$, then $\tilde{C}^{\,0}_{ij}$ contains two independent scalar coefficients functions multiplying $(n_{-\mu}g_{\nu\rho} - n_{-\nu}g_{\mu\rho})/n_-v$ and $(n_{+\mu}g_{\nu\rho} - n_{+\nu}g_{\mu\rho})n_-v$, respectively. The reparameterization-invariance constraints on the coefficient functions discussed in the literature [17, 29, 30] refer to these scalar coefficients. However, for the following considerations it will not be necessary to enumerate all the possible structures in detail. The notation (87) also implies that the matrix elements of the operators on the right-hand side are taken with respect to the full SCET(hc,c,s) Lagrangian, not just the leading-power Lagrangian. For the purposes of power counting and renormalization-group evolution it is sometimes convenient to make explicit the effect of power-suppressed interactions in the form of time-ordered products, however, we shall not use this convention here; the time-ordered product operators are part of the matrix elements with the full SCET(hc,c,s) Lagrangian.

The μ-dependence of the coefficient functions has two origins. One is the scale dependence of the QCD weak currents, unless the current is conserved. The other is related to the factorization of the hard modes and is specific to the effective theory. This dependence is compensated by the dependence of the operators $O^0_j(s)$, $O^{1\mu}_j(s_1,s_2)$ on μ. We implicitly assume a factorization scheme (dimensional regularization with minimal subtractions) that does not introduce a hard cut-off. Then $O^{1\mu}_j(s_1,s_2)$ does not mix into $O^0_j(s)$ under renormalization, since it is λ suppressed relative to $O^0_j(s)$ in SCET(hc,c,s). On the other hand the time-ordered products with power-suppressed SCET(hc,c,s) interactions included in the matrix element of $O^0_j(s)$ may require counterterms proportional to $O^{1\mu}_j(s_1,s_2)$, so the converse is not true. Schematically, we have

$$\mu\frac{d}{d\mu}\begin{pmatrix}\langle O^0_j\rangle \\ \langle O^1_j\rangle\end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\begin{pmatrix}\langle O^0_j\rangle \\ \langle O^1_j\rangle\end{pmatrix} \tag{88}$$

where $\langle\ldots\rangle$ denotes taking an appropriate matrix element.

In [17] there appeared two operators at tree level at order λ with fields $\bar\xi_C A_{\perp C} h_v$, which take the expressions

$$-\bar\xi_C\,\Gamma_i\,\frac{\not{n}_-}{2m_b}\,[iD^\mu_{\perp C}W_C]\,h_v, \qquad -\bar\xi_C\,i\overleftarrow{D}^\mu_{\perp C}\frac{1}{in_+\overleftarrow{D}_C}\frac{\not{n}_+}{2}\Gamma_i W_C\,h_v. \tag{89}$$

Despite their different appearance both structures can be related to (86). If at tree level $\tilde{C}^{1\mu}_{ij}$ has pieces proportional to $\delta(\hat{s}_1)\delta(\hat{s}_2)$, then the corresponding term in (87) collapses to

$$\frac{1}{m_b}\,\bar\xi_C\,[iD^\mu_{\perp C}W_C]\,\Gamma'_j h_v, \tag{90}$$

[11] Without loss of generality we may adopt a frame where $v_\perp = 0$, so $n_+v = 1/n_-v$.

of which the first of the two structures is a linear combination. Pieces proportional to $\delta(\hat{s}_1 - \hat{s}_2)$ result in

$$\frac{1}{m_b}\int_{-\infty}^{0} d\hat{s}\,(\bar{\xi}_C\, i\overleftarrow{D}^{\mu}_{\perp C} W_C)_s \Gamma'_j\, h_v = \frac{1}{n_- v}\,\bar{\xi}_C\, i\overleftarrow{D}^{\mu}_{\perp C}\frac{1}{in_+\overleftarrow{D}_C} W_C\,\Gamma'_j\, h_v, \tag{91}$$

which relates to the second structure. The result (87) given above is also consistent with the analysis of the general operator basis given in [29], simplified to the choice $v_\perp = 0$.

4.2 Definition of the $C_i\,\xi_\pi$ term

We define $\xi_\pi(q^2)$ through the matrix elements of the SCET(hc,c,s) operators $\bar{\xi}_C\,\Gamma'_j h_v$. Since only $\Gamma'_j = 1$ does not vanish between $\langle\pi(p')|$ and $|\bar{B}(p)\rangle$, this defines only one function, independent of the original Dirac matrix Γ_i. The precise definition reads[12]

$$\langle\pi(p')|(\bar{\xi}_C\, W_C\, h_v)(0)|\bar{B}(p)\rangle \equiv 2E\,\xi_\pi(q^2) \tag{92}$$

with $E = (n_- v)(n_+ p')/2 = (m_B^2 - q^2)/(2m_B)$ the energy of the outgoing pion in the $\bar{B}$ meson rest frame. (We neglect $m_\pi^2 \sim \lambda^4$.) By definition $\xi_\pi(q^2)$ includes also dynamical effects at the scale $\sqrt{m_B\Lambda}$ through hard-collinear effects. In (92) collinear expressions include collinear and hard-collinear, and the matrix element is taken with respect to the full SCET(hc,c,s) Lagrangian, not just the leading-power Lagrangian (in which case the matrix element would vanish). As explained above, $\xi_\pi(q^2)$ also depends on a renormalization and factorization scale μ, such that $d\xi_\pi(q^2)/d\mu$ may contain a term proportional to the matrix element of $O_j^{1\mu}(s_1, s_2)$.

The matrix element of the first term on the right-hand side of (87) can be expressed in terms of $\xi_\pi(q^2)$, since one collinear convolution integral can always be trivially done. Choosing $x = 0$,

$$\begin{aligned}
&\langle\pi(p')|\sum_j\int d\hat{s}\,\tilde{C}^0_{ij}(\hat{s}, m_b/\mu)\,O^0_j(s)|\bar{B}(p)\rangle \\
&\quad= \int d\hat{s}\,\tilde{C}^0_{i1}(\hat{s}, m_b/\mu)\,\langle\pi(p')|\,e^{isn_+P}\,(\bar{\xi}_C\,W_C\,h_v)(0)\,e^{-isn_+P}|\bar{B}(p)\rangle \\
&\quad= \int d\hat{s}\,e^{isn_+p'}\,\tilde{C}^0_{ij}(\hat{s}, m_b/\mu)\,\langle\pi(p')|\bar{\xi}_C\,W_C\,h_v|\bar{B}(p)\rangle \\
&\quad= 2E\,C_i(E/m_b, m_b/\mu)\,\xi_\pi(q^2)
\end{aligned} \tag{93}$$

where in the second line P is the momentum operator. In passing to the third line, we neglected $p - m_b v$, since this difference is important only beyond leading power. (The momentum operator in the effective theory on $|\bar{B}(p)\rangle$ gives $p - m_b v$, because the large

[12] We do not distinguish $|\bar{B}(p)\rangle$ from the meson eigenstate $|\hat{B}_v\rangle$ of the leading order HQET Lagrangian, because the difference is a higher-order effect.

component $m_b v$ is scaled out.) In the last line we used (92), and defined the momentum-space coefficient function $C_i(E/m_b, m_b/\mu)$.

A few remarks on the definition of $\xi_\pi(q^2)$ are in order:

(i) The definition is not unique. For instance, it can be redefined by a hard-scattering term. This freedom has been used in [12] to identify $\xi_\pi(q^2)$ with one of the physical form factors. However, for the factorization proof it is more convenient to define $\xi_\pi(q^2)$ as a SCET(hc,c,s) matrix element as done above.

(ii) If we attempt to factorize $\langle\pi(p')|\bar{\xi}_C\, W_C\, h_v|\bar{B}(p)\rangle$ further into light-cone distribution amplitudes, we find that the resulting convolution integrals have endpoint singularities. Furthermore, three-particle light-cone distribution amplitudes of the pion and the B meson would appear already at leading power. This can be seen explicitly from the tree-level matching of the current to SCET(c,s) in the previous section. We also find that $\xi_\pi(q^2)$ scales as λ^3, since as discussed above $\bar{\xi}_C\, W_C\, h_v$ matches onto λ^8 operators in SCET(c,s).

(iii) From the standpoint of factorization one would like to extract the hard-collinear effects from $\xi_\pi(q^2)$ and define a non-perturbative "soft" form factor, which depends only on virtualities of order Λ^2. In practice there is no gain from this, since the main simplification contained in the factorization formula is that there is a single form factor that relates all three pion form factors, up to a standard hard-scattering term. The definition (92) achieves this simplification.

(iv) Our analysis does not allow conclusions on whether $\xi_\pi(q^2)$ is of order λ^3 or $\lambda^3 \times \alpha_s(\sqrt{m_B\Lambda})$. Although at least one hard-collinear spectator interaction is required to turn the soft spectator quark into a collinear quark, the corresponding factor of α_s may be compensated by large logarithms. An analysis of logarithms is needed that we leave for future investigations. From the phenomenological perspective the magnitude of $\xi_\pi(q^2)$ relative to the hard-scattering term is only a numerical question.

Having defined the $C_i\, \xi_\pi$ term in the factorization formula (73), to complete the factorization proof we must show that the matrix element of the second term on the right-hand side of (87) factorizes as

$$\phi_B * T_i * \phi_\pi \tag{94}$$

with convergent convolution integrals of a hard-scattering kernel T_i and light-cone distribution amplitudes.

4.3 Matching on SCET(c,s): Integrating out hard-collinear modes

We compute the matrix element of the second term on the right-hand side of (87),

$$\mathcal{M}_i \equiv \langle\pi(p')|\,\frac{1}{m_b}\,\sum_j\int d\hat{s}_1 d\hat{s}_2\, \tilde{C}^{1\mu}_{ij}\Big(\hat{s}_1,\hat{s}_2,\frac{m_b}{\mu}\Big)\, O^{1\mu}_j(s_1,s_2)|\bar{B}(p)\rangle \tag{95}$$

for $x=0$. Of the three possible Dirac matrices $\Gamma'_j = \{1,\gamma_5,\gamma_\perp^\nu\}$ only $\gamma_\perp^\nu$ can have a non-vanishing matrix element, since there is no external transverse vector available. Hence we

can simplify the previous expression to

$$\frac{1}{m_b}\int d\hat{s}_1 d\hat{s}_2\,\frac{g_\perp^{\mu\nu}}{2}\,\tilde{C}^{1\mu}_{i\nu}\Big(\hat{s}_1,\hat{s}_2,\frac{m_b}{\mu}\Big)\,\langle\pi(p')|(\bar{\xi}_C\,W_C)_{s_1}(W_C^\dagger i\not{D}_\perp W_C)_{s_2}\,h_v|\bar{B}(p)\rangle. \tag{96}$$

Defining the momentum-space coefficient function

$$\frac{g_\perp^{\mu\nu}}{2}\,\tilde{C}^{1\mu}_{i\nu}\Big(\hat{s}_1,\hat{s}_2,\frac{m_b}{\mu}\Big)\equiv\int d\tau_1 d\tau\,e^{-i\tau_1\hat{s}_1}\,e^{-i\tau(\hat{s}_2-\hat{s}_1)}\,C_i^1\Big(\tau;\tau_1,\frac{m_b}{\mu}\Big), \tag{97}$$

and proceeding as in the derivation of (93), we obtain

$$\mathcal{M}_i=\frac{1}{m_b}\int d\tau\,C_i^1\Big(\tau;\frac{E}{m_b},\frac{m_b}{\mu}\Big)\int d\hat{s}\,e^{-i\tau\hat{s}}\,\langle\pi(p')|(\bar{\xi}_C\,W_C)(0)(W_C^\dagger i\not{D}_\perp W_C)(sn_+)\,h_v(0)|\bar{B}(p)\rangle. \tag{98}$$

Note that the second integral defines a function $\Xi_\pi(\tau,q^2)$, so this tells us that before integrating out hard-collinear modes the three form factors can be expressed in terms of two functions $\xi_\pi(q^2)$ and $\Xi_\pi(\tau,q^2)$. Since the second depends on two variables, this is in general not a useful result, unless the three coefficient functions C_i^1 are nearly the same, so that the integral over τ defines the same function. This will be the case in the limit that we neglect hard (but not hard-collinear) quantum corrections.

We now show that the operator $O_j^{1\mu}(s_1,s_2)\sim\bar{\xi}_C A^\mu_{\perp hc}h_v$ matches only on four-quark operators of the form $(\bar{q}_s h_v)(\bar{\xi}_c\xi_c)$ with no additional fields or derivatives. The most general form a SCET(c,s) operator with non-vanishing $\langle\pi|\dots|\bar{B}\rangle$ matrix elements can take is

$$[\text{objects}]\times\Big(\bar{\xi}_c\Gamma'_k h_v\Big)\Big(\bar{q}_s\Gamma'_l\,\{1,\not{n}_+/2\}\,\xi_c\Big). \tag{99}$$

The "objects" can be chosen from

$$\begin{aligned}
&n_-^\mu,\ n_+^\mu,\ g^{\mu\nu},\ \epsilon^{\mu_\perp\nu_\perp\rho\sigma}n_{-\rho}n_{+\sigma},\ \frac{1}{in_-\partial},\ \frac{1}{in_-\partial},\\
&\partial_\perp,\ A_{\perp c},\ A_{\perp s},\ n_+\partial,\ n_+A_s,\ n_-\partial,\ n_-A_c,\\
&\bar{\xi}_c\frac{\not{n}_+}{2}\Gamma'_m\xi_c,\ \bar{q}_s\frac{\not{n}_+}{2}\Gamma'_m q_s,\ \bar{q}_s\frac{\not{n}_-}{2}\Gamma'_m q_s,\ \bar{q}_s\Gamma''_m q_s,
\end{aligned} \tag{100}$$

with Γ''_m a basis for the remaining eight boost-invariant Dirac structures. The notation is symbolic. For instance, if the object is $1/(in_-\partial)$, (99) means that this can act on any combination of soft fields that the operator may contain. The following power counting argument relies on dimensional analysis and boost invariance, so we list the corresponding properties of the objects in Table 2. Note that we cannot use factors of $1/m_b$ or n_-v to build operators, because the hard-collinear loops that are eliminated in the matching onto SCET(c,s) do not depend on these variables. What makes SCET(c,s) power counting non-trivial is the possibility to use $1/(in_-\partial)$, related to the non-localities in the product of soft fields, which in turn is related to the fact that the n_- components of soft momenta are as large as the n_- components of hard-collinear momenta. Since every factor of $1/(in_-\partial)$

Object O	$[\lambda]_O$	boost	$[d]_O$	n_i	Object O	$[\lambda]_O$	boost	$[d]_O$	n_i
$(in_-\partial)^{-1}$	-2	-1	-1	n_1	$\bar{\xi}_c \frac{\not{n}_+}{2}\Gamma'_m \xi_c$	4	-1	3	n_8
$(in_+\partial)^{-1}$	0	$+1$	-1	n_2					
n_-^μ	0	$+1$	0	n_3	$\bar{q}_s \frac{\not{n}_+}{2}\Gamma'_m q_s$	6	-1	3	n_{9a}
n_+^μ	0	-1	0	n_4					
$\partial_\perp$, $A_{\perp c}$, $A_{\perp s}$	2	0	1	n_5	$\bar{q}_s \frac{\not{n}_-}{2}\Gamma'_m q_s$	6	$+1$	3	n_{9b}
$n_+\partial$, n_+A_s	2	-1	1	n_6					
$n_-\partial$, n_-A_c	4	$+1$	1	n_7	$\bar{q}_s \Gamma''_m q_s$	6	0	3	n_{9c}

Table 2: Scaling properties of the building blocks of SCET(c,s) operators. $[l]_O = n$ means that O scales with λ^n. The columns labelled "boost" give the scaling α^n of O under boosts $n_- \to \alpha n_-$, $n_+ \to \alpha^{-1} n_+$. The mass dimension is denoted $[d]_O$. The last column defines the integers n_i that specify the number of occurrences of O in an operator.

decreases the λ scaling of an operator, it is important to constrain the number of times this factor can occur.

Let n_i be the number of times a certain object occurs in an operator of the form (99) as defined in the Table. The λ scaling $[\lambda]$, boost scaling $[\alpha]$ and mass dimension $[d]$ of the operator (99) are then given by

$$[\lambda] \;=\; 10 - 2n_1 + 2n_5 + 2n_6 + 4n_7 + 4n_8 + 6(n_{9a} + n_{9b} + n_{9c}), \tag{101}$$

$$[\alpha] \;=\; -n_1 + n_2 + n_3 - n_4 - n_6 + n_7 - n_8 - n_{9a} + n_{9b}, \tag{102}$$

$$[d] \;=\; 6 - n_1 - n_2 + n_5 + n_6 + n_7 + 3(n_8 + n_{9a} + n_{9b} + n_{9c}). \tag{103}$$

The SCET(hc,c,s) operators we match to SCET(c,s) are boost-invariant, so we impose $[\alpha] = 0$, and solve (102,103) for n_1 and n_2 to obtain

$$[\lambda] \;=\; 4 + [d] - n_3 + n_4 + n_5 + 2n_6 + 2n_7 + 2n_8 + 4n_{9a} + 2n_{9b} + 3n_{9c}, \tag{104}$$

$$n_1 \;=\; 3 - \frac{[d]}{2} + \frac{1}{2}\Big(n_3 - n_4 + n_5 + 2n_7 + 2n_8 + 2n_{9a} + 4n_{9b} + 3n_{9c}\Big), \tag{105}$$

$$n_2 \;=\; 3 - \frac{[d]}{2} + \frac{1}{2}\Big(- n_3 + n_4 + n_5 + 2n_6 + 4n_8 + 4n_{9a} + 2n_{9b} + 3n_{9c}\Big). \tag{106}$$

Since all the n_i are non-negative, the first equation yields $[\lambda] \geq 4 + [d] - n_3$. To turn this into something useful, we need to limit n_3, the number of occurrences of n_-^μ. The possible ways that n_-^μ can appear in (99) are either because there are free Lorentz indices (if the SCET(hc,c,s) operator has such indices) or because it multiplies one of the γ matrices in Γ'_k, Γ'_l, ... The SCET(hc,c,s) operators of interest have only transverse indices (from

$A^{\mu}_{\perp,hc}$ or from $\gamma^{\mu}_{\perp}$ in Γ'_j), so any factor of n^{μ}_{-} actually contracts to 0. The one exception is $\epsilon^{\mu_\perp\nu_\perp\rho\sigma}n_{-\rho}n_{+\sigma}$. Hence $n_3 \leq n_4$ and

$$[\lambda] \geq 4 + [d]. \tag{107}$$

For $O^{1\mu}_j(s_1, s_2) \sim \bar{\xi}_C A^{\mu}_{\perp hc} h_v$ we have $[d] = 4$, so $[\lambda] \geq 8$. Since we are looking for terms of order λ^8, the only solution is $n_5 = \ldots n_{9c} = 0$, which implies that (99) reduces to a four-quark operator with no additional fields or derivatives, and with $n_1 = n_2 = 1$ as was to be shown.

Before we exploit this result it is instructive to reconsider the SCET(hc,c,s) operator $\bar{\xi}_{hc}\Gamma'_j h_v$, which defines $\xi_\pi(q^2)$. Now $[d] = 3$, so $[\lambda] \geq 7$. However, $[\lambda] = 7$ is in fact not possible, since in this case n_1 would have to be half-integer. The leading contributions have $[\lambda] = 8$ and either $n_4 - n_3 = 1$, $n_5 = \ldots = 0$, in which case $n_1 = 1$, $n_2 = 2$ or $n_3 - n_4 = 0$, $n_5 = 1$, $n_6 = \ldots = 0$ in which case $n_1 = n_2 = 2$. The second solution corresponds to operators that can have an additional transverse soft or collinear gluon field. This confirms our earlier conclusion that $\xi_\pi(q^2)$ scales as λ^8, may involve three-particle distributions at leading power, and contains divergent convolution integrals (since larger values of n_1 or n_2 lead to more divergent integrals). In addition we now learn that no four-particle distributions or products of three-particle distributions can appear in $\xi_\pi(q^2)$ at any order in perturbation theory.

As shown above matching the operator $(\bar{\xi}_C\, W_C)(0)(W^\dagger_C i\not{D}_\perp W_C)(sn_+)\, h_v(0) \sim \bar{\xi}_C \not{A}_{\perp hc} h_v$ to SCET(c,s) involves only four-quark operators with one occurrence of $1/(in_-\partial)$ and $1/(in_+\partial)$ each at leading order in λ. In convolution notation, the operator is

$$\int_{-\infty}^{0} ds' \int_0^\infty dt \left[(\bar{\xi}_c W_c)(x_+)\Gamma'_k (Y^\dagger_s h_v)(x_-)\right] \\ \times \left[(\bar{q}_s Y_s)(x_- + tn_-)\Gamma'_l \left\{1, \not{n}_+/2\right\} (W^\dagger_c \xi_c)(x_+ + s'n_+)\right] \tag{108}$$

with $x_\mp$ defined in (25). The operator is local in the transverse position, but non-local along n_+ and n_-. In the following we put $x = 0$. The solution to (104) also implied $n_3 = n_4$. Since $\Gamma'_{k,l}$ do not contain $n_\mp$, this excludes the $\not{n}_+/2$ structure in the curly bracket of (108). This will be important later, because it follows from this that of the two light-cone distribution amplitudes of the B meson, $\phi_{B\pm}$, only ϕ_{B+} appears in the factorization formula. Furthermore, parity and Lorentz invariance limit the possible combinations of Dirac matrices to the three cases $\Gamma'_k = \Gamma'_l$. The colour structure of the operator (108) is $[(\bar{\xi}_c W_c)_a (Y^\dagger_s h_v)_b]\, [(\bar{q}_s Y_s)_b (W^\dagger_c \xi_c)_a]$, because only the combination that makes $\bar{\xi}_c \xi_c$ and $\bar{q}_s h_v$ a colour-singlet will be relevant. The colour indices a, b will be suppressed.

The convolution integrals in (108) are corrected by dimensionless coefficient functions, which can depend on dimensionless ratios of Lorentz invariants. Consider a hard-collinear loop integral with an insertion of the SCET(hc,c,s) operator and with external legs corresponding to the four-quark operator above. The momentum space Feynman rule for the insertion of $\int d\hat{s}\, e^{-i\tau\hat{s}}\, \bar{\xi}_C(0)\not{A}_{\perp\mathrm{hc}}(sn_+)h_v(0)$ is $2\pi\delta(\tau - (n_-v)(n_+p'_3)/m_b)$, where p'_3 denotes the momentum of the hard-collinear gluon. Therefore only the transverse and n_- component of the hard-collinear p'_3 loop integral are performed, while $n_+p'_3$ is kept fixed (and

related to τ by the delta function). The invariants, on which the result of a hard-collinear loop integration can depend, are $(n_+p'_j)(n_-l)$ $(j=1,2,3)$ and μ^2, where p'_1 and p'_2 are the momenta of the external collinear quark lines, and l is the momentum of the soft spectator quark. It is important that the residual momentum of the heavy quark does not appear in this list. The residual momentum can always be routed through the diagram such that it flows from the external heavy quark line directly to the external momentum of the current insertion, so that it never enters the hard-collinear loop propagators. Trading p'_2 for p', the independent dimensionless variables can be chosen as

$$\frac{n_+p'n_-l}{\mu^2},\ \frac{n_+p'_1}{n_+p'},\ \frac{n_+p'_3}{n_+p'}. \tag{109}$$

The coefficient function can depend on powers of logarithms of the first ratio, but it can in principle be an arbitrary function of the second two ratios. In terms of the coordinate space convolutions in (108) n_-l is replaced by t^{-1}, and $n_+p'_1$ is replaced by s^{-1}. Inserting the general coefficient function we obtain the matching relation

$$\begin{aligned}
&\int d\hat{s}\, e^{-i\tau\hat{s}}\,\langle\pi(p')|(\bar{\xi}\,W_c)(0)(W_c^\dagger i\not{D}_\perp W_c)(sn_+)\,h_v(0)|\bar{B}(p)\rangle \\
\longrightarrow\quad &\sum_k\int ds'\int dt\;\tilde{J}_k(m_b\tau/E;n_+p's',\ln(E/(\mu^2 t))) \\
&\times\langle\pi(p')|\Big[(\bar{\xi}_cW_c)(0)\Gamma'_k(Y_s^\dagger h_v)(0)\Big]\Big[(\bar{q}_sY_s)(tn_-)\Gamma'_k(W_c^\dagger\xi_c)(s'n_+)\Big]|\bar{B}(p)\rangle
\end{aligned}\tag{110}$$

valid for the leading power form factors. We used that the delta function from the Feynman rule sets $n_+p'_3$ to $m_b\tau/(n_-v)$.

The SCET(c,s) Lagrangian does not couple collinear and soft fields, so the matrix element formally factorizes

$$\begin{aligned}
&\langle\pi(p')|\Big[(\bar{\xi}_cW_c)(0)\Gamma'_k(Y_s^\dagger h_v)(0)\Big]\Big[(\bar{q}_sY_s)(tn_-)\Gamma'_k(W_c^\dagger\xi_c)(s'n_+)\Big]|\bar{B}(p)\rangle \\
&\quad= -(\Gamma'_k)_{\alpha\alpha'}(\Gamma'_k)_{\beta\beta'}\,\langle\pi(p')|(\bar{\xi}_{c\alpha}W_c)(0)(W_c^\dagger\xi_{c\beta'})(s'n_+)|0\rangle \\
&\qquad\times\langle 0|(\bar{q}_{s\beta}Y_s)(tn_-)(Y_s^\dagger h_{v\alpha'})(0)|\bar{B}(p)\rangle.
\end{aligned}\tag{111}$$

Potential subtleties are related to the convergence of the convolution integrals, an issue that we address below. The two matrix elements define the light-cone distribution amplitudes of the pion and the B meson given in (61). These definitions coincide with the standard definition of the distribution amplitudes. Inserting them into (111) we obtain the two traces

$$\mathrm{tr}\left(\frac{1+\not{v}}{2}\not{n}_\pm\gamma_5\Gamma'_k\frac{\not{n}_-}{2}\gamma_5\Gamma'_k\right)\tilde{\phi}_{B\pm}(t) \tag{112}$$

multiplied by the position-space B meson distribution amplitude. The trace for the minus sign vanishes, because $\not{n}_-$ commutes or anti-commutes with $\gamma_5\Gamma'_k$, so $\tilde{\phi}_{B-}(t)$ does not appear in the final result. The momentum space B meson distribution amplitude is defined by

$$\tilde{\phi}_{B+}(t)\equiv\int_0^\infty d\omega\, e^{-i\omega t}\,\phi_{B+}(\omega). \tag{113}$$

Using this and (61) in (111), the right-hand side of (110) turns into

$$\frac{if_\pi}{4}\, n_+p'\, \frac{if_B m_B}{4}\, n_-v \sum_k \mathrm{tr}\left(\frac{1+\not{v}}{2}\not{n}_+\gamma_5\Gamma'_k\frac{\not{n}_-}{2}\gamma_5\Gamma'_k\right) \int_0^\infty d\omega \int_0^1 du\, \phi_{B+}(\omega)\, \phi_\pi(u)$$
$$\times \int ds' \int dt\, e^{-i\omega t}\, e^{i(1-u)s'n_+p'}\, \tilde{J}_k(m_b\tau/E; n_+p's', \ln(E/(\mu^2 t)) \qquad (114)$$

The second line defines the momentum-space hard-collinear coefficient function

$$\int ds' \int dt\, e^{-i\omega t}\, e^{i(1-u)s'n_+p'}\, \tilde{J}_k(m_b\tau/E; n_+p's', \ln(E/(\mu^2 t))$$
$$\equiv \frac{1}{\omega\, n_+p'}\, J_k(m_b\tau/E; u, \ln(E\omega/\mu^2)). \qquad (115)$$

With this definition J_k is a series of powers of $\ln\omega$ given that $\tilde{J}_k$ contained only powers of $\ln t$. Finally, inserting (115) into (114), (114) into (110), and (110) into (98) we obtain

$$\mathcal{M}_i = \frac{1}{m_b}\frac{if_\pi}{4}\frac{if_B m_B}{4}\, n_-v \sum_k \mathrm{tr}\left(\frac{1+\not{v}}{2}\not{n}_+\gamma_5\Gamma'_k\frac{\not{n}_-}{2}\gamma_5\Gamma'_k\right)$$
$$\times \int_0^\infty d\omega\, \frac{\phi_{B+}(\omega)}{\omega} \int_0^1 du\, \phi_\pi(u) \int d\tau\, C_i^1\left(\tau; \frac{E}{m_b}, \frac{m_b}{\mu}\right) J_k(m_b\tau/E; u, \ln(E\omega/\mu^2)). \qquad (116)$$

This has the form of the hard-scattering term in (73) and (94), where we can now identify the kernels T_i with the convolutions of a hard coefficient function with the hard-collinear coefficient functions, $T_i \sim \sum_k C_i^1 \star J_k$.

Thus we have shown that the three form factors can be represented in terms of s single function $\xi_\pi(q^2)$ plus a hard-scattering term, which is a convolution of a short-distance kernel with the leading twist two-particle light-cone distribution amplitudes of the pion and the B meson. To complete the proof, we must show that the convolution integrals converge. The crucial point is that boost invariance constrained the coefficient functions J_k to contain powers of $\ln\omega$ only. The soft convolution integrals are therefore the moments

$$\frac{1}{\lambda_B^{(n)}} \equiv \int_0^\infty d\omega\, \frac{\phi_{B+}(\omega)}{\omega}\, \ln^n\left(\frac{m_B\omega}{\mu^2}\right), \qquad (117)$$

which generalize the moment $\lambda_B = \lambda_B^{(0)}$ introduced in [2]. The integrals converge provided $\phi_{B+}(\omega) \sim \omega^a$ with $a > 0$ for $\omega \to 0$ and $\phi_{B+}(\omega) \sim \omega^b$ with $b < 0$ for $\omega \to \infty$. In the Wandzura-Wilczek approximation one finds that $a = 1$ and $\phi_{B+}(\omega) = 0$ for ω larger than a critical value [31] implying convergence, as do recent QCD sum rule estimates [32]. Furthermore, if the integrals (117) converge for the distribution amplitude evaluated at one scale μ_0, it converges for all scales [33]. It seems safe to conclude that the moments (117) exist. This would not be the case for the analogous moments of $\phi_{B-}(\omega)$, at least within the Wandzura-Wilczek approximation [12], so the absence of $\phi_{B-}(\omega)$ from the factorization formula is crucial.

There is no analogous constraint on the functional form of the u-integrand, since J_k can be an arbitrary function of u. However, we argued in Section 2 that an endpoint singularity of a collinear integral implies that soft and collinear contributions must be factorized explicitly, so a collinear endpoint divergence should match to a soft endpoint divergence to make the sum of all terms unambiguous. But the soft convolution integral is convergent, so the u-integral must be convergent, too.[13] There is no need to be concerned about the convergence of the τ-integral. This integral is a convolution of the hard matching coefficient with the hard-collinear matching coefficient, and originates in the matching of QCD to SCET(hc,c,s). If divergent, the integral could be regularized perturbatively, and the coefficients C_i^1 and J_k would be defined accordingly. In any case, we can always imagine matching QCD directly to SCET(c,s), in which case the integral $\sum_k C_i^1 \star J_k$ is obtained directly as the coefficient function. This concludes the proof of the factorization formula.

The arguments presented here carry over to the case of transitions to light flavour-non-singlet vector mesons without essential modifications. For vector mesons the matrix elements of $\bar{\xi}_C W_C \Gamma_j' \, h_v$ are non-zero for $\Gamma_j' = \{\gamma_5, \gamma_\perp^\mu\}$, so instead of (92) we have two form factors commonly denoted as $\xi_\perp(q^2)$ and $\xi_\|(q^2)$. The conclusion that the SCET(hc,c,s) current operator with the hard-collinear gluon field matches to a four-quark operator that factorizes into convergent convolutions of light-cone distribution amplitudes remains valid, so it follows that the seven independent vector meson form factors reduce to two form factors plus the hard-scattering term as expected [12, 13].

4.4 Comparison with previous work

The insights gained from the details of the matching of the heavy-to-light current to SCET(c,s) and the structure of the factorization proof result in a picture that differs in several respects from previous work. We briefly mention these differences.

Motivated by the suggestion [13] that symmetries in large-energy effective theory and heavy quark effective theory reduce the number of independent form factors at large recoil energy, the factorization formula (1,73) was suggested in [12] and shown to be valid at the level of one-gluon exchange. It was noted that formally sub-leading twist-3 two-particle light-cone distribution amplitudes yield leading-power contributions and it was shown that these can be absorbed into the definition of the universal form factor(s). The possibility that three-particle amplitudes also contribute at leading power was not considered. The present work has shown that this is in fact the case, but again these contributions can be absorbed into the definition of the universal form factor(s).

Subsequent work discussed the $C_i \, \xi_\pi$ term in the factorization formula in the framework of SCET [16] and investigated whether some relations between form factors implied by the factorization formula could be valid including power corrections [17]. These works proceeded on the assumption that the leading-power current in SCET(hc,c,s) is $\bar{\xi}_C \Gamma h_v$. From [15] and the present work we know that formally sub-leading currents are also needed.

[13] A potential collinear endpoint divergence could not be related to the $\xi_\pi(q^2)$ term in the factorization formula, since with the definition of $\xi_\pi(q^2)$ as a matrix element in SCET(hc,c,s) adopted here, soft-collinear factorization is never performed for this term.

Here we have shown that these currents match to conventional hard-scattering contributions in SCET(c,s), so that they do not affect the discussion of the $C_i\,\xi_\pi$ term. On the other hand, the suppression of the matrix element of the formally leading current implies that the investigation of power corrections to the leading term is difficult, and hence that the conclusions of [17] should be revisited. We leave this for the future, but the present work has shown that power corrections are suppressed by $1/m_b$ (and not, perhaps, $1/\sqrt{m_b}$), a point that was left open in [17].

The most detailed discussion of the factorization formula for the form factors in the context of SCET is provided by [15], where in particular the factorization formula has already been claimed to be shown. In addition to mentioning the suppression of the leading-power current this paper also suggests to prove the factorization formula by two-step matching of QCD to SCET(hc,c,s) to SCET(c,s), a line of argument that we followed in the present work. However, in our opinion the factorization arguments of [15] are incomplete or incorrect in three crucial aspects. All of them have to do with the fact that the second matching step to SCET(c,s) was not considered in sufficient detail. The first point concerns the claim that the form factor scales as λ^3. Since the power counting of SCET(c,s) fields and states has not been worked out, the statement appears to follow from the power counting of the hard-scattering term that can be obtained by inspection, and by comparing this to the power counting of other non-factorizable terms. Since this comparison is done for the time-ordered products in SCET(hc,c,s), but without explicit matching to SCET(c,s), it is not clear whether all time-ordered products that scale equally in SCET(hc,c,s) give equal contributions to the form factors. The second point concerns the definition of "factorizable" and "non-factorizable" terms. The definition in [15] is based on the formal distinction of SCET(hc,c,s) operators according to whether they contain only dressed soft quark fields after the decoupling of soft gluons ("factorizable operators") or also dressed soft gluon fields ("non-factorizable"). This should be contrasted with the point of view taken in this paper that the distinction of factorizable and non-factorizable terms is related to soft-collinear factorization in SCET(c,s), such that non-factorizable operators are those whose matrix elements do not factorize naively into soft and collinear matrix elements. Comparing the former definition to ours, it is not clear to us that the term T_0^F identified as "factorizable" in [15] matches only to a four-quark operator without endpoint divergences, since the corresponding time-ordered product defined in SCET(hc,c,s) contains the leading power SCET(hc,c,s) collinear interactions that may match to three-particle terms in SCET(c,s). Finally, although the paper states that the convolution integrals that arise from matching factorizable time-ordered products to SCET(c,s) are convergent, this claim is not substantiated at any point.

5 Conclusion

To summarize, we have shown that heavy-to-light form factors at large recoil energy of the light meson factorize according to $F_i = C_i\,\xi_\pi + \phi_B \star T_i \star \phi_\pi$ to all orders in α_s, and at leading power in an expansion in $1/m_b$. The main point of the formula is that a larger

number of form factors in QCD reduces to a smaller number of form factors ξ_π (similar to the reduction of the number of heavy-to-heavy form factors in heavy quark effective theory) up to a hard-scattering term (similar in structure to the pion form factor at large momentum transfer). The coefficient functions C_i contain hard short-distance effects (scale m_b), while the T_i represent convolutions of hard and hard-collinear (scale $\sqrt{m_B\Lambda}$) effects. All quantities have been defined in the framework of soft-collinear effective theory. Because the matrix elements of the formally leading currents vanish, the factorization formula for the form factors is actually a statement about power-suppressed effects.

The more complicated factorization of the heavy-to-light form factors compared to $B \to D$ form factors and light-meson form factors is related to the interplay of soft and collinear physics in a heavy-to-light transition. This results in multi-particle contributions and "endpoint singularities" at leading power in the heavy quark expansion, if one attempted to factorize the form factors along the lines of the pion form factor. In this paper we showed that the terms that cause these complications can be absorbed into the $C_i\,\xi_\pi$ term and then showed that the remainder factorizes in the standard way. This allowed us to by-pass a detailed treatment of endpoint singularities.

We believe, however, that the present paper provides a first step towards understanding endpoint singularities in the framework of effective field theory, since we identified their origin in the structure of soft-collinear factorization within SCET(c,s). While soft and collinear interactions are formally factorized in the SCET(c,s) Lagrangian, this separation requires in general an additional regularization, which corresponds to the regularization of endpoint singularities. We illustrated how this works for a toy integral, but at present it is not clear how to implement the concept of soft-collinear factorization in general and in practice. A viable framework to perform calculations, when endpoint singularities cannot be so easily disposed of as in the factorization proof for heavy-to-light form factors, is also necessary to understand how to sum large logarithms of different scales in the hard-scattering kernel T_i. We did not address this important point here. Finally, we mention that some of our conclusions can be adapted to other scattering processes such as the pion form factor at sub-leading power, where endpoint singularities are also expected to appear. For instance, Figure 2 has a straightforward translation to this case, when we replace "soft" and "collinear" by the two types of collinear modes (describing the clusters of energetic, nearly massless particles moving in opposite directions in an appropriate reference frame) relevant for light-meson form factors. We therefore expect interesting extensions of our work into different directions.

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